Robustness and Dynamic Sentiment *

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Abstract

Errors in survey expectations display waves of pessimism and optimism and significant sluggishness. This paper develops a novel theoretical framework of time-varying beliefs capturing these empirical facts. In our model, dynamic beliefs arise endogenously due to agents' attitude toward alternative models. Decision-maker's distorted beliefs generate countercyclical risk aversion, procyclical portfolio weights, countercyclical equilibrium asset returns, and excess volatility. A calibrated version of our model is shown to match salient features in equity markets.

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A large empirical literature in macroeconomics and finance documents significant dynamic variation in forecasters' expectational errors or belief distortions. One recurring empirical finding is that these distortions oscillate between periods of pessimism and optimism and respond to economic shocks. Figure 1 replicates this finding for two measures of economic activity (GDP growth and unemployment) and inflation where we plot the difference between the consensus survey forecast for each macroeconomic variable and a model forecast. We highlight two observations from these series. First, confirming earlier work, we document waves of both pessimism and optimism across all variables. Second, a lesser known empirical fact is that belief distortions display significant sluggishness: the large increase in pessimism after the 2008 Great Financial Crisis only dissipates very slowly, remaining in a pessimistic state for a prolonged period.[1]

Time-variation in belief distortions has spawned a large theoretical literature based on belief-based models of diagnostic expectations (Bordalo, Gennaioli, Ma, and Shleifer, 2020), fading memory (Nagel and Xu, 2019), or skewed priors (Afrouzi and Veldkamp, 2019). Preference-based models rely on ambiguity aversion (Ilut and Schneider, 2014) or robust control (Bhandari, Borovička, and Ho, 2019; Hansen, Szőke, Han, and Sargent, 2020). In this paper, we contribute to the latter strand of literature. We present a novel theoretical framework where agents' attitude toward alternative models endogenously generates time-varying pessimism and optimism and agents' belief distortions react sluggishly. We then study its implications for asset pricing and match salient features of equity markets using a calibrated model.

The seminal work of Hansen and Sargent (2001) posits that economic agents with an aversion to model ambiguity seek robustness by entertaining a family of models constructed as a neighborhood around a baseline model and optimize against the worst-case within this family.[2] In a Bayesian interpretation, the worst-case model that the decision-maker guards against can be viewed as representing endogenously distorted pessimistic beliefs.[3] Throughout the literature on Hansen-Sargent robustness, Kullback and Leibler (1951) divergence (also known as relative entropy) is used to measure the discrepancy between

[1] We verify the degree of sluggishness using regressions of belief distortions (defined as the difference between the consensus survey forecast and a model forecast) on past distortions and the state of the economy:

$$\text{distortion}_t = \text{constant} + \beta \times \text{distortion}_{t-1} + \gamma \times \text{state of the economy}_{t-1} + \epsilon_t,$$

where we proxy the state of the economy by inflation or GDP growth. t-Statistics of estimated coefficients for β range between 5.39 (inflation) and 10.42 (GDP growth).

[2] See Hansen and Sargent (2008) for a textbook treatment.

[3] Alternatively, the family of models being considered by the decision-maker can be viewed as the set of non-unique priors in the max-min expected utility of Gilboa and Schmeidler (1989). Motivated by these authors, the work of Chen and Epstein (2002), Epstein and Schneider (2003) and related papers offers another approach to modeling ambiguity aversion in a dynamic setting. Maccheroni, Marinacci, and Rustichini (2006a,b) show that the framework of variational preferences nests these different approaches. Strzalecki (2011) characterizes multiplier preferences axiomatically. Most recently, Hansen and Sargent (2020) construct a continuous-time extension of variational preferences that combines ambiguity aversion in the sense of Chen and Epstein (2002) with model uncertainty aversion in the sense of Anderson, Hansen, and Sargent (2003).

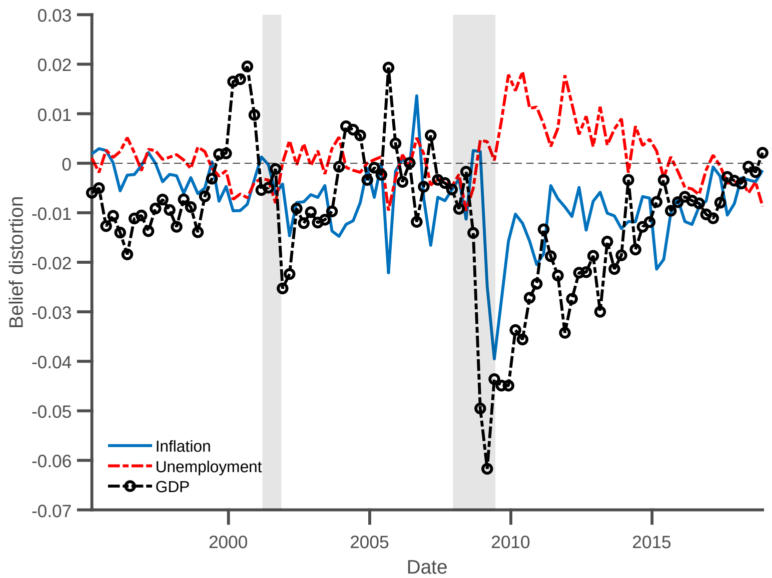


Figure 1. **Belief Distortions: Difference between Subjective and Objective Beliefs**

Notes: This figure plots a proxy for belief distortions for real GDP growth, inflation, and unemployment defined as the difference between the mean one-year-ahead forecasts from Consensus Economics and corresponding statistical (objective) forecasts from a VAR(2). Gray shaded bars indicate recessions as defined by the NBER. Data is quarterly and running from 1995 to 2018.

models. To match the time-variation in belief distortions uncovered in the data, we replace relative entropy by the family of Cressie and Read (1984) divergences and construct a discrepancy measure which preserves recursivity and homotheticity.

We summarize our main theoretical contributions as follows. First, while relative entropy has strong foundations in information theory, econometric theory, and often offers tractability for dynamic problems, it leads to myopic belief distortions in a setting with i.i.d. shocks. Belief distortions that are independent of economic shocks and constant over time are difficult to reconcile with empirical evidence. In our model, Cressie-Read divergences render the discrepancy measure state-dependent via a unique sentiment state variable that summarizes past shocks and belief distortions. As a result, even in an environment with i.i.d. shocks, subjective beliefs change endogenously in response to changes in the economy.

The Cressie-Read family is parameterized by a parameter η and nests relative entropy as a special case when η tends to unity. Intuitively, the parameter η governs the desire for intertemporal smoothness of belief distortions. Values of η close to unity indicate a high desire for intertemporal smoothing for belief distortions, leading to strong sluggishness. As η moves away from unity, the endogenous dynamics of belief distortions depend on whether η is greater or smaller than 1. When $\eta < 1$, agent's subjective beliefs display momentum-like dynamics and become less pessimistic or more optimistic after positive shocks, leading to procyclical sentiment. For $\eta > 1$, agent's subjective beliefs feature reversal-like dynamics and turn more pessimistic or less optimistic after positive shocks. Moreover, the volatility of sentiment also responds to economic shocks and does so asymmetrically between positive and negative shocks. When $\eta < 1$, pessimistic sentiment volatility shrinks after positive shocks and increases after negative shocks. The effect is opposite for $\eta > 1$.

Second, we examine the implications of the Cressie-Read divergence for portfolio choice and equilibrium asset pricing. We first characterize the agent's preferences with Cressie-Read divergence in the robust control framework of Hansen and Sargent. The state-dependence of the Cressie-Read divergence produces dynamic risk aversion (a stochastic variance multiplier in the language of stochastic differential utility à la Duffie and Epstein (1992)). The elasticity of dynamic risk aversion (the variance multiplier) with respect to the sentiment state variable is exactly $1 - \eta$. When $\eta < 1$, risk aversion is countercyclical, leading to a procyclical portfolio weight. The opposite result holds for $\eta > 1$. The agent's dynamic belief distortion also generates intertemporal hedging demands, which grow with the investment horizon. Moreover, to accommodate optimism as well as pessimism, we embed a model of optimism inspired by Bhandari, Borovička, and Ho (2019) in a regime-switching setting. Alternating waves of pessimism and optimism magnify the state-dependence of portfolio choice.

We further study a general equilibrium model for a representative agent with Epstein and Zin (1989) preferences and Cressie-Read divergence. To this end, we calibrate our model from an empirical measure of time-varying pessimism and optimism extracted from a large cross-section of forecast data on aggregate economic activity. Using GDP survey data from Consensus Economics, we define a measure of sentiment as the difference between the one-year ahead consensus forecast and a one-year ahead forecast from a VAR(2). The calibrated model features rich dynamics and illustrates that time-varying sentiment leads to equity risk premia and Sharpe ratios that quantitatively match the data. We also explore the effect of pessimism and optimism separately in the calibrated model and find that equity premia and volatilities are higher (lower) and risk-free rates are lower (higher) in states of pessimism (optimism). Monte Carlo simulations reveal that the distributions of the equilibrium quantities are skewed and heavy-tailed in bad states of nature, reflecting the higher volatility of beliefs in those states of the world.

Finally, as a theoretical contribution, we show that our construction of the intertemporal Cressie-Read divergence measure satisfies recursivity and constitutes a recursive ambiguity index in the sense of Maccheroni, Marinacci, and Rustichini (2006b), thereby guaranteeing time-consistency of preferences. In addition, we also preserve homotheticity of the resulting preferences, which benefits the tractability of the approach. We motivate and derive our definition of the Cressie-Read divergence measure in a discrete-time setting and we prove that relative entropy is the unique dynamic divergence measure that satisfies recursivity. A (unique) stochastic scaling factor is introduced for our Cressie-Read divergence measure to maintain the same recursivity. Interestingly, we find that the stochastic scaling factor that is instrumental in guaranteeing recursivity and time-consistency is a power function of the sentiment state variable driving the nonlinear dynamics of the model.

Related Literature: Our paper builds on the robust control literature studying pessimistic subjective beliefs such as Hansen and Sargent (2001), Anderson, Hansen, and Sargent (2003), and Hansen, Sargent, Turmuhambetova, and Williams (2006), among many others. Different from this literature, which imposes an entropy penalty, we show that for values $\eta \neq 1$, the Cressie-Read divergence family generates time-varying beliefs and risk aversion. Moreover, our model calibration shows that time-varying pessimism and optimism induces rich dynamics in asset prices.

The two papers closest to ours are Bhandari, Borovička, and Ho (2019) and Hansen, Szőke, Han, and Sargent (2020). Bhandari, Borovička, and Ho (2019) study the effects of pessimism and optimism on macroeconomic fluctuations in the presence of nominal rigidities and labor market frictions to address the unemployment volatility puzzle. Our paper is significantly different from their paper along several dimensions. First, they use entropy-based robust control and allow for exogenous variation in the strength of the preference for robustness. We, in contrast, generate dynamic sentiment endogenously. Second, they build a macroeconomic model with frictions, while we consider frictionless markets and focus on asset pricing implications. Hansen, Szőke, Han, and Sargent (2020) construct twisted relative entropy in order to generate countercyclical concerns for model misspecification and corresponding risk premia. Our specification based on the Cressie-Read divergence can be seen as an alternative mechanism where these effects are obtained endogenously and within the framework of Maccheroni, Marinacci, and Rustichini (2006b), thus satisfying dynamic consistency by construction. Szőke (2019) estimates the model of Hansen, Szőke, Han, and Sargent (2020) using survey data.

Chamberlain (2020) offers an excellent survey on empirical methods for robust portfolio choice as an example of econometric issues in decision making. He shows how dynamic $\phi-$divergence preferences can be used for purposes of sensitivity analysis when an investor fears misspecification. We complement his analysis by solving explicitly for optimal portfolio choice and dynamic general equilibrium, as well as calibrating the model based on empirical estimates of pessimism and optimism from survey data.

On the empirical side, we follow the literature that elicits subjective beliefs from survey data. For example, Bianchi, Ludvigson, and Ma (2020) compare survey responses to machine-learning based predictions about macroeconomic quantities and find strong evidence for time-varying pessimism and optimism in the data. Adam, Matveev, and Nagel (2021) study different surveys and reject the hypothesis that respondents are always pessimistically biased towards expected returns as predicted by the Hansen and Sargent model and argue that they are often optimistic.

Motivated by empirical and experimental evidence on belief formation, a large literature in behavioral economics and finance proposes models of deviations from rational expectations. Prominent examples include diagnostic expectations as, e.g., in Bordalo, Gennaioli, and Shleifer (2018), Bordalo, Gennaioli, Porta, and Shleifer (2019), and Bordalo, Gennaioli, Ma, and Shleifer (2020), and extrapolative expectations, see, e.g., Hong and Stein (1999), Barberis, Greenwood, Jin, and Shleifer (2015), Barberis (2018), Jin and Sui (2019), and Li and Liu (2019). Our model endogenously generates similar belief dynamics, in the sense that belief distortions are driven by a state variable that summarizes past belief distortions and fundamental shocks experienced by the decision-maker. The belief dynamics displays momentum-like features when the Cressie-Read parameter η is less than 1 and contrarian behavior when η is larger than 1. While our specification does not introduce the recency bias documented in the extrapolative expectation literature, it does introduce asymmetric responses after positive and negative fundamental shocks due to their different impact on belief volatility. The endogenous response of belief volatility to fundamental shocks differentiates our model from diagnostic expectations where the diagnostic distribution has a constant variance in an AR(1) model, see, e.g., Bordalo, Gennaioli, and Shleifer (2018).

Another related strand of the literature studies Cressie-Read divergence measures to estimate subjective beliefs or stochastic discount factors. For example, Chen, Hansen, and Hansen (2021) bound the divergence between subjective beliefs and their rational expectations benchmark using $\phi-$divergences. Similarly, using convex duality, Korsaye, Trojani, and Quaini (2020) estimate minimum-divergence stochastic discount factors in markets with frictions. Different from this literature, which does not impose any preference structure and does not study portfolios in general equilibrium, we derive optimal portfolios in a Merton (1969) setting and study the effect of time-varying sentiment on equilibrium quantities.

Our paper is also related to different strands of the literature that study time-varying beliefs or risk-aversion using alternative mechanisms. For example, Bidder and Dew-Becker (2016), Collin-Dufresne, Johannes, and Lochstoer (2016), Dew-Becker and Nathanson (2019), and Kozlowski, Veldkamp, and Venkateswaran (2020), among many others, generate rich belief dynamics endogenously in asset pricing models with simple fundamentals when agents are allowed to learn. Asset pricing studies that feature time-varying risk aversion include the habit models of Constantinides (1990), Detemple and Zapatero (1991), and Campbell and Cochrane (1999) where time-varying risk aversion is tightly linked to the level of consumption relative to its recent past history. While in these models time-varying risk aversion is exogenously imposed on the utility function of the representative agent, in our setting, stochastic risk aversion arises endogenously due to the agent's concern for model misspecification.

Outline of the paper: The rest of the paper is organized as follows. Section 1 provides the theoretical framework and studies the robust utility index. Section 2 examines the partial equilibrium portfolio problem. Section 3 develops a general equilibrium model with an Epstein and Zin (1989) representative agent with Cressie-Read divergence. Section 4 estimates empirical proxies of time-varying sentiment that we use in Section 5 for model calibration. Finally, Section 6 concludes. To save space, we collect all proofs and further technical details in a separate appendix.

1 The Model

1.1 The Cressie-Read Divergence

We first motivate our definition of the Cressie-Read divergence in a discrete-time setting. Consider a state process y whose dynamics under the baseline model $\mathbb{B}$ and an alternative model $\mathbb{U}$ follow

$$y_{t+\Delta t} = Ay_t + \sigma\sqrt{\Delta t}\epsilon_t^{\mathbb{B}} \quad \text{and} \quad y_{t+\Delta t} = Ay_t + \sigma\big(\sqrt{\Delta t}\epsilon_t^{\mathbb{U}} - u_t\Delta t\big),$$

respectively, where $t = m\Delta t$ with $m = 0, 1, 2, \ldots,$ A, σ are constants, and $\{\epsilon_t^{\mathbb{B}}\}$ and $\{\epsilon_t^{\mathbb{U}}\}$ are i.i.d. standard normal noise. The alternative model $\mathbb{U}$ is parameterized by a process u, which we call the belief distortion. For a positive u_t, the mean of $y_{t+\Delta t}$ conditioning on y_t is $\sigma u_t \Delta t$ less under $\mathbb{U}$ than under $\mathbb{B}$, hence, the agent is pessimistic under the alternative model $\mathbb{U}$. The likelihood ratio of the distribution for y_t between $\mathbb{U}$ and $\mathbb{B}$ is $Z_t = \prod_{s=\Delta t}^{t} N_s$, where $Z_0 = 1$ and $N_{t+\Delta t} = e^{-\sqrt{\Delta t}u_t\epsilon_t^{\mathbb{B}} - \frac{1}{2}u_t^2\Delta t}$ is the likelihood ratio of the conditional distribution for $y_{t+\Delta t}|y_t$ between $\mathbb{U}$ and $\mathbb{B}$.

Given a divergence function ϕ, convex and with $\phi(1) = 0$, $\phi(Z_t)$ describes the discrepancy between $\mathbb{U}$ and $\mathbb{B}$ on $\mathcal{F}_t$. We introduce the ϕ-divergence as

$$R^{\mathbb{U}} = \mathbb{E}^{\mathbb{B}}\Big[\textstyle\sum_{t=0}^{\infty}\beta^t\big(\phi(Z_{t+\Delta t}) - \phi(Z_t)\big)\Big], \tag{1}$$

where β is the agent's subjective discounting factor and $\phi(Z_{t+\Delta t}) - \phi(Z_t)$ measures the increment of the discrepancy from time t to $t + \Delta t$.[4] Using a second-order Taylor expansion and Girsanov theorem, we obtain

$$R^{\mathbb{U}} = \tfrac{1}{2}\mathbb{E}^{\mathbb{B}}\Big[\textstyle\sum_{t=0}^{\infty}\beta^t\phi''(Z_t)Z_t^2u_t^2\Delta t\Big] + o(\Delta t) = \tfrac{1}{2}\mathbb{E}^{\mathbb{U}}\Big[\textstyle\sum_{t=0}^{\infty}\beta^t\phi''(Z_t)Z_tu_t^2\Delta t\Big] + o(\Delta t). \tag{2}$$

Time consistency. The ϕ-divergence in (2) is evaluated at time 0. Its (conditional) value at time t can be similarly defined to measure the conditional discrepancy from time t onward:

$$R_t^{\mathbb{U}} = \tfrac{1}{2}\mathbb{E}_t^{\mathbb{U}}\Big[\textstyle\sum_{s=t}^{\infty}\beta^{s-t}\phi''(Z_{t,s})Z_{t,s}u_s^2\Delta t\Big] + o(\Delta t), \tag{3}$$

[4] Equation (1) shows that it is without loss of generality to restrict alternative models $\mathbb{U}$ to be absolutely continuous with respect to $\mathbb{B}$, otherwise, both Z_t and $\phi(Z_t)$ could be infinite with positive probability, in which case $R^{\mathbb{U}}$ would be ill-defined. Note that because absolute continuity requires fixing σ in the continuous-time model below, we already impose the same restriction in the discrete-time setting here.

where $\mathbb{E}_t^{\mathbb{B}}[\cdot]$ is the conditional expectation and $Z_{t,s} = Z_s/Z_t$ is the conditional likelihood ratio. To ensure time consistency of the optimization problem considered later, we need the ϕ-divergence process $\{R_t^{\mathbb{U}}\}$ to be recursive:

$$R_0^{\mathbb{U}} = \tfrac{1}{2}\mathbb{E}^{\mathbb{U}}\left[\textstyle\sum_{s=0}^{t-1}\beta^s\phi''(Z_s)Z_s u_s^2\Delta t\right] + \mathbb{E}^{\mathbb{U}}\left[\beta^t R_t^{\mathbb{U}}\right] + o(\Delta t). \tag{4}$$

However, this property does not automatically hold unless ϕ satisfies $\phi''(Z)Z = $ constant, which uniquely pins down ϕ as the Kullback and Leibler (1951) divergence (relative entropy). In this case,

$$R_t^{\mathbb{U}} = \mathbb{E}_t^{\mathbb{U}}\left[\tfrac{1}{2}\textstyle\sum_{s=t}^{\infty}\beta^{s-t}u_s^2\Delta t\right] + o(\Delta t),$$

which is the entropy divergence introduced by Hansen and Sargent (2001). Therefore the entropy divergence is the unique dynamic divergence satisfying (4).

For a general ϕ, in order to maintain (4), we need to extend the definition in (3) by introducing a scaling factor Φ_t:

$$R_t^{\mathbb{U}} = \tfrac{1}{2\Phi_t}\mathbb{E}^{\mathbb{U}}\left[\textstyle\sum_{s=t}^{\infty}\beta^{s-t}\phi''(Z_{t,s})Z_{t,s}u_s^2\Delta t\right] + o(\Delta t). \tag{5}$$

In this paper, we consider ϕ belonging to the family of Cressie and Read (1984) divergences:

$$\phi(z) = \frac{1-\eta+\eta z - z^\eta}{\eta(1-\eta)}, \quad \eta \in \mathbb{R}\setminus\{0,1\}.^{5} \tag{6}$$

The function ϕ is convex, satisfies $\phi(1) = 0$, is decreasing when $z \in (0,1)$, and increasing when $z > 1$.[6] Note that the Cressie-Read divergence family includes several well-known divergence functions: with $\eta = 1$ being the KL divergence, $\eta = 0$ is known as Burg (1972) entropy, $\eta = 1/2$ corresponds to the Hellinger (1909) distance, and $\eta = 2$ describes the modified χ^2-divergence.

For a Cressie-Read divergence function ϕ, the unique choice of Φ_t that ensures the recursivity property (4) to be satisfied by $\{R_t^{\mathbb{U}}\}$ in (5), is

$$\Phi_t = Z_t^{1-\eta}. \tag{7}$$

It immediately follows that $\frac{1}{\Phi_t}\phi''(Z_{t,s})Z_{t,s} = Z_s^{\eta-1}$, and hence,

$$R_t^{\mathbb{U}} = \tfrac{1}{2}\mathbb{E}_t^{\mathbb{U}}\left[\textstyle\sum_{s=t}^{\infty}\beta^{s-t}Z_s^{\eta-1}u_s^2\Delta t\right] + o(\Delta t). \tag{8}$$

Theorem 2 in Maccheroni, Marinacci, and Rustichini (2006b) implies that our construction

[5]For $\eta \in \{0,1\}$, the function $\phi(z)$ is defined as the corresponding limit.

[6]The affine component in ϕ does not impact our discrepancy measure. Rather it ensures positivity of ϕ and $\phi(1) = 0$ for any η. Because of the affine component our specification avoids the monotonically decreasing cases of other Cressie-Read parameterizations, see Chen, Hansen, and Hansen (2020). Furthermore, our objective function and optimization problem are entirely different from theirs.

of $R^{\mathbb{U}}$ yields a recursive ambiguity index which leads to time consistency of preferences. Notice that the scaling factor Φ in (7) is instrumental in guaranteeing recursivity and time-consistency and leads to the state-dependent weight $Z_s^{\eta-1}$ before u_s^2 in (8) that drives the nonlinear dynamics of the model.

Continuous-time limit. Motivated by the discrete-time example, we work with a continuous-time setting in the rest of the paper. Let $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{B})$ be a standard probability space and $B^{\mathbb{B}}$ is a d-dimensional Brownian motion. Probability measure $\mathbb{B}$ is the agent's baseline model and a family of alternative models $\mathbb{U}$ is parameterized by a $\mathbb{R}^d$-valued bounded belief distortion process u. For each given bounded u, $\mathbb{U}$ is defined as

$$\frac{d\mathbb{U}}{d\mathbb{B}}\Big|_{\mathcal{F}_T} = Z_T,^{7} \quad \text{where} \quad Z_t = \exp\Big(-\int_0^t \tfrac{1}{2}|u_s|^2 ds - \int_0^t u_s' dB_s^{\mathbb{B}}\Big), \quad t \in [0,T], \tag{9}$$

and u' is the transpose of u. For every random outcome $\omega \in \Omega$, the density $Z_T(w)$ describes the change of measure for this outcome under $\mathbb{U}$ compared to $\mathbb{B}$. Larger $Z(\omega)$ implies more weight is put on ω under the alternative model $\mathbb{U}$. Denote $Z_{t,s} = Z_s/Z_t$ for any $0 \le t \le s \le T$ as the conditional density. Conditional probabilities are determined via Bayes' formula.[8]

Following the motivation from the discrete-time example, we introduce the following (conditional) *Cressie-Read divergence* to measure the discrepancy between $\mathbb{U}$ and $\mathbb{B}$ on $[t,T]$:

$$R_t^{\mathbb{U}} = \frac{1}{\Phi_t}\mathbb{E}_t^{\mathbb{B}}\Big[\int_t^T e^{-\delta(s-t)}\Psi_s dD_{t,s}\Big], \tag{10}$$

where δ is a constant subjective discount rate and $D_{t,s} = \phi(Z_{t,s})$ measures the realized divergence between $\mathbb{U}$ and $\mathbb{B}$ on $[t,s]$, and Φ is the scaling factor ensuring the recursivity of $\{R_t^{\mathbb{U}}\}$. Extending the discrete-time definition, we introduce a positive stochastic weight Ψ in (10) to ensure optimization problems considered later remain homothetic.

When Φ is chosen as (7), Lemma 1 in Appendix B shows that $R^{\mathbb{U}}$ in (10) is transformed to

$$R_t^{\mathbb{U}} = \frac{1}{2}\mathbb{E}_t^{\mathbb{U}}\Big[\int_t^T e^{-\delta(s-t)}\Psi_s Z_s^{\eta-1}|u_s|^2 ds\Big] \tag{11}$$

and is recursive. Equation (11) underscores three important implications for the dynamics of subjective beliefs. First, it is immediate to see that in the entropy case, i.e., when $\eta = 1$, the divergence measure simply integrates the discounted squared belief distortions u. Second, when $\eta \neq 1$, the Cressie-Read divergence introduces a weight $Z^{\eta-1}$. Recall from (9) that Z is state-dependent: it not only reflects the history of belief distortions u but also the realizations of the Brownian shocks $B^{\mathbb{B}}$. Therefore, Z can be seen as the cumulative belief distortion, which aggregates different belief distortions and Brownian shocks across time and across

[7]Because u is bounded, $\mathbb{E}^{\mathbb{B}}[Z_T] = 1$, therefore $\mathbb{U}$ is a probability measure equivalent to $\mathbb{B}$.

[8]For an event $A \in \mathcal{F}_s, \mathbb{P}^{\mathbb{U}}(A|\mathcal{F}_t) = \mathbb{E}_t^{\mathbb{B}}[Z_{t,s}1_A]$, for any $t \le s$. Here $1_A(\omega) = 1$ if $\omega \in A$, 0 otherwise, is the indicator for A.

states. We call Z the sentiment state variable. In the optimization problems considered later, Z is the crucial state variable that links the agent's future belief distortion to historic ones. This feature allows us to capture the sluggishness in belief distortions documented in Figure 1. Third, because the Cressie-Read divergence in (11) uses a power of Z, $Z^{\eta-1}$, as a weight for $|u|^2$, states with larger values of $Z^{\eta-1}$, for a given value of u, contribute more to the Cressie-Read divergence $\{R_t^U\}$ than states with smaller values of $Z^{\eta-1}$. This implies that deviating from $\mathbb{B}$ on states with larger values of $Z^{\eta-1}$ is more costly. Intuitively, we can think of the Cressie-Read divergence as an extended entropy divergence with a dynamic belief- and state-dependent weight $Z^{\eta-1}$, which depends endogenously on the past distortion u. As shown later, this mechanism is key to generate endogenous and dynamic beliefs and sentiment. In particular, when $\eta < 1$, the agent's sentiment improves (she is less pessimistic or more optimistic) as fundamentals improve, and vice versa for $\eta > 1$.

Before moving to the agent's utility, we present a technical modification of equation (11), which helps pin down boundary conditions for our numerical analysis for portfolio choice and general equilibrium problems studied later. More specifically, we consider an approximation of (11)

$$R_t^{\mathbb{U}} = \frac{1}{2}\mathbb{E}_t^{\mathbb{U}}\Big[\int_t^T e^{-\delta(s-t)}\Psi_s Z_{s\wedge\tau}^{\eta-1}|u_s|^2 ds\Big], \tag{12}$$

where $\tau = \inf\{t \geq 0 : Z_t \leq \underline{z} \text{ or } Z_t \geq \overline{z}\}$, with constants $0 < \underline{z} < \overline{z}$. The interval $[\underline{z}, \overline{z}]$ contains all plausible adjustments, so that $Z < \underline{z}$ ($Z > \overline{z}$) is regarded as unreasonably underweighted (overweighted) in our model. The constant $\underline{z}$ (resp. $\overline{z}$) is fixed to be sufficiently close to zero (resp. large) so that only extreme weights are excluded and the probability of $\tau < T$ can be made arbitrarily small. When $t > \tau$, the weight is frozen at $Z_\tau^{\eta-1}$ and $R_t^{\mathbb{U}}$ can be viewed as an entropy divergence.

1.2 The Utility Index with Pessimism and Optimism

The main mechanism behind the dynamic and state-dependent subjective beliefs driving the key results in our paper can be understood from an analysis of the utility index process. To this end, we first consider pessimism and optimism separately, and then build a regime-switching model combining both.

Pessimism: To model pessimism, we fix a parameter $\theta^P > 0$, which measures the strength of the preference for robustness. Consider a consumption stream c and a utility function U. We then define a utility index $\mathcal{U}^{P,c}$ as the *pessimistic utility* for c:

$$\mathcal{U}_t^{P,c} = \inf_u \mathbb{E}_t^{\mathbb{U}}\Big[\int_t^T e^{-\delta(s-t)}\delta U(c_s)ds + e^{-\delta(T-t)}\epsilon U(c_T) + \frac{1}{\theta^P}R_t^{\mathbb{U}}\Big], \tag{13}$$

where ϵU, with a positive constant ϵ, is the bequest utility and $\{R_t^{\mathbb{U}}\}$ is given in equation (12). The utility index is the minimal value among utilities under a family of alternative models

$\mathbb{U}$, indexed by the belief distortion u. The minimization with respect to u addresses the agent's concern for model misspecification. The minimizer u^* is called the worst-case belief distortion. Under its associated subjective beliefs, the expected utility of c is so low that even after adding $R_t^{\mathbb{U}}/\theta^P$, it still achieves the minimal value on the right-hand side of equation (13). The role of $\{R_t^{\mathbb{U}}\}$ is to constrain the choice of u by penalizing distortions that are too large and deemed unreasonable.

To understand the intuition of problem (13), let us consider an example where c follows $\frac{dc_t}{c_t} = \mu_c dt + \sigma_c' dB_t^{\mathbb{B}}$ for a constant μ_c and a constant vector σ_c. Because (12) indicates that $R^{\mathbb{U}}$ depends on the state Z when $\eta \neq 1$, we consider $\mathcal{U}^{P,c}$ as a function of the state variables c and Z, i.e., $\mathcal{U}_t^{P,c} = \mathcal{U}(t, c_t, Z_t)$ for some function $\mathcal{U}$. The dynamic programming principle implies that u^* satisfies

$$u_t^* = \frac{\theta^P \Gamma_t [1 + E_t]}{\Psi_t} Z_{t\wedge\tau}^{1-\eta}, \tag{14}$$ [9]

where $\Gamma_t = \partial_Z \mathcal{U}(-Zu^*) + \partial_c \mathcal{U}(c\sigma_c)$ is the instantaneous volatility of the pessimistic utility $\mathcal{U}^{P,c}$ and $E_t = \frac{Z_t}{\Gamma_t} \partial_Z \Gamma_t$ (with the division calculated component-wise) is the elasticity of Γ_t with respect to Z_t.

When $\eta \neq 1$, the choice of u impacts the dynamics of Z, which in turn affects the value of $\mathcal{U}^{P,c}$ and its volatility. Therefore, the optimization takes into account the utility volatility Γ and its sensitivity E with respect to Z in determining the worst-case belief in (14). In the entropy case, with $\eta = 1$, $\mathcal{U}$ and Γ are independent of Z, hence $E \equiv 0$ and $u^* = \frac{\theta^P \Gamma_t}{\Psi_t}$. The belief distortion for $\eta \neq 1$ involves two important additional factors, both of which reflect the endogenous sentiment state variable Z in (12). The direct effect of Z on the optimal belief distortion u^* in (14) is nonlinear and depends on the Cressie-Read parameter $\eta \neq 1$, as discussed below. In addition, the indirect effect through E acknowledges that Z also impacts the utility volatility Γ.

More generally, when c is non-Markovian, the characterization in (14) remains valid, and is presented in Lemma 2 of the Online Appendix. In this case, problem (13) can be characterized by a Forward Backward Stochastic Differential equation (FBSDE), where the forward component describes the dynamics of Z, the backward component represents the optimization problem (13), and the two components are coupled via u^*. This extends the framework of stochastic differential utility of Duffie and Epstein (1992) as well as the generalized stochastic differential utility introduced in Lazrak and Quenez (2003), which are characterized as BSDEs. In addition to the technical complications stemming from the presence of the forward component Z, the extension is important from an economic point of

[9] The drift of the process $e^{-\delta t} Z_t \mathcal{U}(t, c_t, Z_t) + \int_0^t e^{-\delta s} Z_s \big[\delta U(c_s) + \frac{1}{2\theta^P} \Psi_s Z_{s\wedge\tau}^{\eta-1} |u_s|^2\big] ds$, divided throughout by $e^{-\delta t} Z_t$, contains the following terms depending on u:

$$\tfrac{1}{2} \partial_{ZZ}^2 \mathcal{U}(-Z_t u_t)^2 + \partial_{Zc}^2 \mathcal{U}(-Z_t u_t)(c_t \sigma_c) - u_t \big[\partial_Z \mathcal{U}(-Z_t u_t) + \partial_c \mathcal{U}(c_t \sigma_c)\big] + \tfrac{1}{2\theta^P} \Psi_t Z_t^{\eta-1} |u_t|^2.$$

The minimization problem over u in (13) and the dynamic programming principle imply that u^* minimizes the previous expression. Then the first-order condition in u yields (14).

view, since the endogenous forward component is precisely the mechanism that generates dynamic belief distortions endogenously.[10]

Using the worst-case belief distortion u^* in (14), we obtain the following representation for the pessimistic utility $\mathcal{U}^{P,c}$:

$$\mathcal{U}_t^{P,c} = \mathbb{E}_t^{\mathbb{B}}\left[e^{-\delta(T-t)}\epsilon U(c_T) + \int_t^T e^{-\delta(s-t)}\delta U(c_s)\,ds - \int_t^T e^{-\delta(s-t)}\frac{\theta^P}{2\Psi_s}Z_{s\wedge\tau}^{1-\eta}|\Gamma_s|^2\left(1-|E_s|^2\right)ds\right]. \quad (15)$$

A heuristic way of understanding the economic mechanism of this representation builds on the analysis of Duffie and Epstein (1992). In their terminology, utility variance enters the utility index as a penalty. In (15), the penalty is determined by applying the variance multiplier $\frac{\theta^P}{2\Psi}Z^{1-\eta}$ to the variance of the utility index $|\Gamma|^2$ adjusted by the squared elasticity of the utility volatility $|E|^2$. The dependence of the variance multiplier on Z captures the endogenous change in risk aversion due to distorted beliefs.[11] The weight factor $Z^{1-\eta}$ introduces belief- and state-dependence in the variance multiplier, which distinguishes it from the standard entropy case. Note that the elasticity of the variance multiplier with respect to Z is exactly $1-\eta$. Therefore $1-\eta$ measures the elasticity of state-dependence in the variance multiplier. To build intuition, we discuss the properties of $\mathcal{U}^{P,c}$ in the following Proposition.

Proposition 1 (Pessimistic utility)**.**

1. *A component of u^* is positive if and only if the corresponding component of $\Gamma[1+E]$ is positive. For a fixed and positive $\Gamma[1+E]$, all components of u^* increase as $Z^{1-\eta}$ increases.*
2. *When a component of u^* is positive, positive shocks to the corresponding component in $B^{\mathbb{B}}$ decrease Z, hence decrease $Z^{1-\eta}$ when $\eta < 1$, or increase $Z^{1-\eta}$ when $\eta > 1$.*

The first result shows that the agent is pessimistic in the worst-case belief compared to the reference belief. To see this most easily, we focus on a one-dimensional case, i.e., $d = 1$. The utility volatility Γ can be interpreted as the sensitivity of $\mathcal{U}^{P,c}$ with respect to the fundamental shocks $dB^{\mathbb{B}}$. From the first result it follows that when $\Gamma[1+E]$ is positive, as is shown in our applications later, the worst-case belief distortion u^* is positive as well. Under $\mathbb{U}$, $dB_t^{\mathbb{B}} = -u_t^* dt + dB_t^{\mathbb{U}}$ where $B^{\mathbb{U}}$ is a Brownian motion under $\mathbb{U}$. Therefore, the expected growth of $B^{\mathbb{B}}$ under $\mathbb{U}$ is underestimated relative to the growth under $\mathbb{B}$, i.e., the agent is pessimistic under $\mathbb{U}$.

The second result explains how shocks affect the belief distortion, which in turn impacts the variance multiplier, risk aversion, and belief volatility. Importantly, the effect crucially hinges on whether η is greater than or smaller than 1. Let us first consider $\eta < 1$.

[10] When either $\eta = 1$ or when Z is an exogenous state variable whose dynamics do not depend on u, then the utility index $\mathcal{U}^{P,c}$ is a generalized stochastic differential utility, characterized by a BSDE. When $\eta \neq 1$, $\mathcal{U}^{P,c}$ is characterized by a FBSDE.

[11] The risk aversion captured by the variance multiplier is in addition to any risk aversion already encoded in $U(c)$, or more generally, in the terminology of Duffie and Epstein (1992), its "aggregator."

Positive fundamental shocks to $B^{\mathbb{B}}$ decrease $Z^{1-\eta}$ due to the second result in Proposition 1. This decreases the variance multiplier $\frac{\theta^P}{2\Psi}Z^{1-\eta}$, thereby lowering the agent's risk aversion. Moreover, by the first result, the worst-case belief distortion u^* decreases as $Z^{1-\eta}$ decreases, making the agent less pessimistic following positive fundamental shocks. Furthermore, from (9), we see that u^* is also the volatility of the logarithm of the state variable Z. Therefore, positive fundamental shocks reduce the volatility of $\log(Z)$ and make the state variable temporarily less sensitive to future fundamental shocks. By the same intuition, negative fundamental shocks to $B^{\mathbb{B}}$ exacerbate the agent's pessimism and also increase the volatility of $\log(Z)$. As a result, the state variable becomes temporarily more sensitive to future fundamental shocks. The asymmetric response after positive and negative fundamental shocks is the key mechanism to generate skewness in equilibrium quantities later.

In contrast, when $\eta > 1$, the previous results are reversed. Positive fundamental shocks to $B^{\mathbb{B}}$ increase $Z^{1-\eta}$, hence increase u^*, i.e., the agent becomes more pessimistic, while negative fundamental shocks tend to reduce pessimism.

In summary, the agent chooses future belief distortions based on the current Z, which is a sufficient statistic of past belief distortions and fundamental shocks experienced. The pessimistic utility exhibits countercyclical risk aversion, pessimism, and volatility of beliefs when $\eta < 1$, while $\eta > 1$ delivers procyclicality. Table 1 summarizes these results.

Table 1. **Responses to Fundamental Shocks $B^{\mathbb{B}}$ in Case of Pessimism**

$\Delta B^{\mathbb{B}}$	Optimal Distortion	Risk Aversion	Sentiment	Volatility Sentiment
		$\eta < 1$		
positive	$\downarrow$	$\downarrow$	less pessimistic	$\downarrow$
negative	$\uparrow$	$\uparrow$	more pessimistic	$\uparrow$
		$\eta > 1$		
positive	$\uparrow$	$\uparrow$	more pessimistic	$\uparrow$
negative	$\downarrow$	$\downarrow$	less pessimistic	$\downarrow$

Another way of understanding the intuition for our results is to view the Cressie-Read penalty function as reflecting a preference of Nature (the fictitious malevolent agent in the max-min expected utility interpretation) for intertemporal smoothing of the process for the belief distortions u^*. The closer η is to 1, the stronger the preference for intertemporal smoothing, and the more inelastic the variance multiplier is with respect to changes in the sentiment state variable. As η moves away from unity, the belief distortions are more dynamic and the sign of the elasticity of the variance multiplier depends on whether η is greater than or smaller than 1. When $\eta < 1$, the decision-maker expects more adverse distortions from Nature in bad times and less adverse distortions in good times; the agent's subjective beliefs therefore exhibit "momentum-like" dynamics, and pessimism gets worse in bad times, while beliefs improve following positive shocks.

In contrast, when $\eta > 1$, the agent expects less adverse distortions in bad times and more adverse distortions following positive shocks. The agent could be seen as thinking that "lightning never strikes twice" after an adverse shock, i.e., if Nature has just used its ammunition in hitting the agent with a negative shock, it will not do so again immediately. Put differently, now the agent's subjective beliefs exhibit reversal-like dynamics.

Optimism: We model optimism in an analogous way to Bhandari, Borovička, and Ho (2019) by considering a constant $\theta^O < 0$. The *optimistic utility* $\mathcal{U}^{O,c}$ for the consumption stream c is defined as

$$\mathcal{U}_t^{O,c} = \sup_u \mathbb{E}_t^{\mathbb{U}}\Big[\int_t^T e^{-\delta(s-t)}\delta U(c_s)ds + e^{-\delta(T-t)}\epsilon U(c_T) + \frac{1}{\theta^O}R_t^{\mathbb{U}}\Big]. \tag{16}$$

The maximization represents the agent's desire to explore the best-case scenario and the maximizer u^* is called the best-case belief distortion. Under $\mathbb{U}$, the expected utility of c is so high that, even after penalizing by $R_t^{\mathbb{U}}/\theta^O$ with $\theta^O < 0$, it still achieves the maximal value on the right-hand side of (16). As in the case of pessimism, $\{R_t^{\mathbb{U}}\}$ continues to penalize belief distortions that are deemed unreasonable.

The characterization of the optimal belief distortion u^* in (14) still holds, with θ^P replaced by θ^O. Therefore, when $\Gamma[1+E]$ is positive in a component (as is the case in applications later), the best-case belief distortion u^* is negative in the same component, due to the negativity of θ^O. As a result, the Brownian motion $B^{\mathbb{B}}$, which follows the dynamics $B_t^{\mathbb{B}} = -u_t^* dt + dB_t^{\mathbb{U}}$ under $\mathbb{U}$, has higher expected growth under $\mathbb{U}$ than under $\mathbb{B}$, i.e., the best-case belief $\mathbb{U}$ is optimistic relative to the reference belief $\mathbb{B}$. We rewrite equation (15) as follows

$$\mathcal{U}_t^{O,c} = \mathbb{E}_t^{\mathbb{B}}\Big[e^{-\delta(T-t)}\epsilon U(c_T) + \int_t^T e^{-\delta(s-t)}\delta U(c_s)\,ds - \int_t^T e^{-\delta(s-t)}\frac{\theta^O}{2\Psi_s}Z_{s\wedge\tau}^{1-\eta}|\Gamma_s|^2\big(1-|E_s|^2\big)\,ds\Big], \tag{17}$$

where the variance multiplier $\frac{\theta^O}{2\Psi}Z^{1-\eta}$ is negative and indicates risk-seeking behavior.[12] The third term on the right-hand side of (17), with a negative variance multiplier, represents the premium or reward the agent adds to the expected utility due to optimism.[13]

We can now state the results of Proposition 1 for the optimistic case.

Proposition 2 (Optimistic utility)**.**

1. *A component of u^* is negative if and only if the corresponding component of $\Gamma[1+E]$ is*

[12]As before, the attitude towards risk expressed by the variance multiplier ought to be viewed in combination with the risk aversion already encoded in $U(c)$. In the case of optimism, the risk-seeking attitude acts to reduce overall risk aversion. For exposition purposes we find it useful to refer to the risk-seeking attitude injected by the variance multiplier as being uncertainty-seeking, while still referring to the overall combined effect as concerning risk aversion.

[13]Heath and Tversky (1991) provide experimental evidence that agents prefer to bet on more ambiguous events when they consider themselves knowledgeable, moreover, they even pay a significant premium to bet on their judgements.

positive. For a fixed and positive $\Gamma[1+E]$, all components of u^* decrease as $Z^{1-\eta}$ increases.

2. *When a component of u^* is negative, positive shocks to the corresponding component in $B^{\mathbb{B}}$ increase Z, hence, increase $Z^{1-\eta}$ when $\eta < 1$, or decrease $Z^{1-\eta}$ when $\eta > 1$.*

The first result shows that the agent is more optimistic in the best-case belief than the reference belief. Moreover, the dynamics of optimism, risk attitude, and belief volatility follow from combining the two results above. When $\eta < 1$, positive fundamental shocks to $B^{\mathbb{B}}$ make the best-case belief distortion u^* even more negative, i.e., optimism becomes more pronounced, the agent becomes more uncertainty-seeking (and as a result less overall risk-averse), and volatility of the logarithm of Z increases as well. As with pessimism, these results are reversed for $\eta > 1$.

1.3 A Regime-Switching Model between Pessimism and Optimism

In the previous section, while the degree of pessimism or optimism in the agent's beliefs depends dynamically on $Z^{1-\eta}$, the agent is either optimistic or pessimistic. To allow for both sentiment states to be present, we introduce a regime-switching model in which the agent changes between optimism and pessimism dynamically. The idea is that when the agent feels less pessimistic, she is more likely to switch to the optimistic state; conversely, when the agent feels less optimistic, she is more likely to switch back to the pessimistic state. In other words, the intensities of switching between states also depend on the state variable Z.

To this end, let I be a process taking values in $\{O, P\}$ indicating whether the agent is in the optimistic or pessimistic state. Let $\nu = \inf\{s \geq t : I_s \neq I_t\}$, i.e., the first time when the agent switches between optimism and pessimism after time t. Given a consumption stream c, we introduce agent's utility index conditional on the state. For the pessimistic utility,

$$V_t^{P,c} = \inf_u \mathbb{E}_t^{\mathbb{U}}\Big[\int_t^{T\wedge\nu} e^{-\delta(s-t)}\Big(\delta U(c_s) + \frac{1}{2\theta^P}\Psi_s Z_{s\wedge\tau}^{\eta-1}|u_s|^2\Big)ds + e^{-\delta(T-t)}\epsilon U(c_T)\,1_{\{\nu>T\}} + e^{-\delta(\nu-t)}V_\nu^{O,c}1_{\{\nu\leq T\}}\Big]. \tag{18}$$

In the previous equation, $V_t^{P,c}$ is the agent's utility at time t, if the agent is pessimistic at time t. It equals the minimization on the right-hand side, which incorporates the discounted intertemporal utility, the Cressie-Read penalty, and the discounted bequest utility if the agent remains pessimistic until the terminal time, or the discounted optimistic utility $V_\nu^{O,c}$ if the agent switches to optimism before the terminal time. When the agent is optimistic at time t, $V_t^{O,c}$ can be defined similarly with $V^{O,c}$ and $V^{P,c}$ swapped in (18), θ^P replaced by θ^O, and the infimum over u changed to supremum.

We model I as a continuous-time Markov chain, independent of $B^{\mathbb{B}}$, with transition intensity λ^O from the state P to O, and the intensity λ^P from O back to P. We assume

$$\lambda_t^O = \Lambda^O(Z_t^{1-\eta}) \quad \text{and} \quad \lambda_t^P = \Lambda^P(Z_t^{1-\eta}), \tag{19}$$

for two decreasing functions Λ^O and Λ^P. Therefore the transition intensities depend on $Z^{1-\eta}$, i.e., the same state variable driving the agent's local belief distortion in the optimistic and pessimistic state as described in the previous subsection.

To build intuition, we focus on the $\eta < 1$ case. Suppose the agent starts in the pessimistic state. Negative fundamental shocks to $B^{\mathbb{B}}$ increase the value of $Z^{1-\eta}$, hence, the agent becomes more pessimistic. Meanwhile, because Λ^O is a decreasing function, the transition intensity λ^O to the optimistic state decreases. Therefore the agent is less likely to become optimistic after bad fundamental shocks. On the other hand, positive fundamental shocks decrease the value of $Z^{1-\eta}$, hence, the agent is less pessimistic. At the same time, the transition intensity λ^O to the optimistic state increases. Consequently, the agent becomes increasingly likely to switch to the optimistic state after experiencing positive fundamental shocks. Once the agent switches to the optimistic state, continuing positive fundamental shocks increase $Z^{1-\eta}$ and the agent becomes more optimistic. At the same time, the transition intensity λ^P decreases, so that the likelihood to switch back to the pessimistic state shrinks.

In summary, in our regime-switching model, the agent's beliefs are dynamic and endogenously driven by a power function of the agent's cumulative distorted belief Z, which summarizes the agent's past belief distortion and history of fundamental shocks.

2 Portfolio Choice

We now have all the ingredients to study the implications of time-varying sentiment on asset prices. Before focussing on asset prices in equilibrium, however, we build intuition in a partial equilibrium setting in the context of optimal portfolio choice.

2.1 The Consumption and Portfolio Choice Problem

Consider a capital market with a risk-free bond with a constant interest rate r and d risky assets whose prices follow

$$dS_t = \text{diag}(S_t)(\mu dt + \sigma dB_t^{\mathbb{B}}),$$

where μ is a constant d-dimensional vector representing expected returns, σ is a constant $d \times d$-matrix describing the return volatilities, and diag(S) is a d-dimensional diagonal matrix with diagonal elements $\{S^1, \ldots, S^d\}$. The agent invests her wealth in the risky assets based on a vector of portfolio weights π and consumes at a rate c. The dynamics of wealth follow

$$dW_t = \big[rW_t + W_t\pi_t'(\mu - r) - c_t\big]dt + W_t\pi_t'\sigma dB_t^{\mathbb{B}}. \tag{20}$$

Depending on the initial sentiment state, the agent's optimal portfolio choice problem is

$$V_0^O = \sup_{\pi,c} V_0^{O,c} \quad \text{or} \quad V_0^P = \sup_{\pi,c} V_0^{P,c}. \tag{21}$$

The agent's optimal consumption and investment strategies are shown to depend on her sentiment state. Problem (21) can also be formulated at time $t \in [0,T]$. We denote their optimal values by V_t^O and V_t^P, respectively.

In (21), we choose CRRA utility $U(c) = \frac{c^{1-\gamma}}{1-\gamma}$ with coefficient of relative risk aversion $0 < \gamma \neq 1$ for both intertemporal and bequest utilities.[14] To maintain homotheticity of problem (21), we follow Maenhout (2004) and choose Ψ in (18) and its optimistic analogue as follows:

$$\Psi_t = (1-\gamma)V_t^P, \text{ if } I_t = P \quad \text{or} \quad \Psi_t = (1-\gamma)V_t^O, \text{ if } I_t = O. \tag{22}$$

Take the pessimistic state for example, combining equations (18) and (22) yields

$$V_t^P = \sup_{\pi,c} \inf_u \mathbb{E}_t^{\mathbb{U}}\Big[\int_t^{T\wedge\nu} e^{-\delta(s-t)}\Big(\delta U(c_s) + \frac{1-\gamma}{2\theta^P} V_s^P Z_{s\wedge\tau}^{\eta-1}|u_s|^2\Big)ds + e^{-\delta(T-t)}\epsilon U(c_T)\, 1_{\{\nu>T\}} + e^{-\delta(\nu-t)}V_\nu^O 1_{\{\nu\leq T\}}\Big].$$

Because the value V^P shows up on both sides of the previous problem, it can be considered as an optimization problem for a (generalized) stochastic differential utility.

2.2 A Two-Stage Example

To understand the impact of the Cressie-Read divergence on the agent's portfolio choice, we now consider a simplified two-stage problem. We simplify the model in two ways. First, we focus on pessimism and rule out optimism. At the end of this subsection, we then explain the impact of optimism and regime-switching. Second, throughout this two-stage example, we freeze the state variable Z as follows. Stage 1 starts from time 0 and ends at time 1, stage 2 starts from time 1 until ∞. In both stages, Z is frozen and only updated once, namely at time 1, i.e., $Z_t = Z_0$ when $t \in [0,1)$; $Z_t = Z_1$ when $t \geq 1$. To simplify further, we consider the case of one risky asset, i.e., $d = 1$. We also set $\underline{z} = 0$ and $\overline{z} = \infty$ so that $\tau = \infty$. We solve the two-stage problem by backward induction.

Second Stage: In the second stage, the agent's optimal consumption and investment problem is

$$V_t^P = \sup_{\pi,c} \inf_u \mathbb{E}_t^{\mathbb{U}}\Big[\int_t^{\infty} e^{-\delta(s-t)}\Big(\delta U(c_s) + \frac{1-\gamma}{2\theta^P} V_s^P Z_1^{\eta-1}|u_s|^2\Big)ds\Big], \quad t \geq 1, \tag{23}$$

[14]The case of log utility is discussed in Appendix C.

subject to (20). Notice that the agent does not update her cumulative belief distortion Z after time 1. Therefore, problem (23) is equivalent to Maenhout (2004) for an entropy penalty, but with robustness preference parameter $\theta^P Z_1^{1-\eta}$ instead of θ^P. The optimal portfolio weight and the worst-case belief distortion are therefore

$$\pi_t = \frac{\mu - r}{\sigma^2} \frac{1}{\gamma + \theta^P Z_1^{1-\eta}} \quad \text{and} \quad u_t = \frac{\mu - r}{\sigma} \frac{\theta^P Z_1^{1-\eta}}{\gamma + \theta^P Z_1^{1-\eta}}, \quad t \geq 1. \tag{24}$$

It follows from equation (24) that the agent's effective risk aversion is $\gamma + \theta^P Z_1^{1-\eta}$, where $Z_1 = \exp(-\frac{1}{2}|u_0|^2 - u_0 B_1^{\mathbb{B}})$. We can now study the effects of η on the agent's risk aversion.

Let $\eta < 1$ and $u_0 > 0$.[15] After positive return shocks to $B_1^{\mathbb{B}}$, Z_1 decreases, and as a result the agent's effective risk aversion $\gamma + \theta^P Z_1^{1-\eta}$ decreases as well (due to $\eta < 1$), which increases the agent's optimal portfolio weight π in the second stage. Meanwhile, the agent's belief distortion u decreases, generating a less pessimistic expected return $\mu - \sigma u$ under the worst-case subjective belief $\mathbb{U}$. In summary, after favorable return shocks in the first stage, the agent becomes less pessimistic and less risk-averse. Negative return shocks lead to the opposite: the investor becomes more pessimistic and more risk-averse.

Under sufficiently good market conditions, $Z_1^{1-\eta}$ tends to $\underline{z}^{1-\eta} = 0$. The agent's effective risk aversion gets close to its minimal value γ and the optimal u collapses to 0. This implies that the agent is no longer pessimistic in the limit. This can also be seen from equation (23) where the Cressie-Read penalty becomes extremely large for nonzero u, due to the large weight $Z_1^{\eta-1}$, so that it is very costly for the agent to deviate from the reference measure $\mathbb{B}$. Under very unfavorable market conditions, $Z_1^{1-\eta}$ explodes to infinity, resulting in an infinitely risk-averse investor who shuns the risky asset. In this case, $u = \frac{\mu - r}{\sigma}$, which leads to a zero equity risk premium under the agent's extremely pessimistic subjective view.

When $\eta > 1$, however, the agent's effective risk aversion $\gamma + \theta^P Z_1^{1-\eta}$ increases after positive return shocks, with the investor reducing her portfolio weight in the risky asset. Meanwhile, u increases and the agent becomes more pessimistic after positive return shocks.

First Stage: Because the agent's optimal strategies in the second stage depend on the realization of Z_1, it is reflected in the value function V_1^P at time 1.[16] The agent's problem in the first stage can now be written as

$$V_0^P = \sup_{\pi, c} \inf_u \mathbb{E}^{\mathbb{U}} \Big[\int_0^1 e^{-\delta t} \Big(\delta u(c_t) + \frac{1 - \gamma}{2\theta^P} V_t^P Z_0^{\eta-1} |u_t|^2 \Big) dt + e^{-\delta} V_1^P \Big]. \tag{25}$$

Here $Z_0^{\eta-1} = 1$ and the continuation utility at time 1 is state-dependent, making the problem (25) state-dependent as well. The agent has an intertemporal hedging demand in the first stage against the (perceived) market condition fluctuation due to her changing belief at time

[15] We verify the latter condition numerically in the fully dynamic model in the next section.

[16] More specifically, $V_1^P = \frac{W^{1-\gamma}}{1-\gamma} e^{f_1^P(Z_1)}$ with $f_1^P(Z_1) = \gamma \log(\gamma \delta^{\frac{1}{\gamma}}) - \gamma \log \Big(\delta + (\gamma - 1) r + \frac{\gamma - 1}{2} \frac{(\mu - r)^2}{\sigma^2} \frac{1}{\gamma + \theta^P Z_1^{1-\eta}} \Big)$.

1. When $\eta = 1$, the continuation utility is state-independent, and problem (25) is equivalent to a problem with an entropy penalty where the optimal strategy is myopic.

Combining Optimism with Pessimism: We now discuss how to introduce optimism and regime-switching in the two-stage model. In the two-stage model, it is natural to assume that the transition between the optimistic and pessimistic states can only happen at time 1. If the agent starts with a pessimistic belief in stage 1, there is a probability $p_1^P = \exp(-\Lambda^O(Z_1^{1-\eta}))$ to remain pessimistic in the second stage. When $\eta < 1$, positive return shocks in the first stage decrease p_1^P. Hence the agent is more likely to be optimistic in the second stage. If the agent starts with an optimistic belief, there is a probability $p_1^O = \exp(-\Lambda^P(Z_1^{1-\eta}))$ she remains optimistic in stage 2. In this case, positive return shocks increase p_1^O when $\eta < 1$, making the agent less likely to switch to pessimism in the second stage.

Allowing for optimism requires $0 < \underline{z} < \overline{z} < \infty$ and Z_1 to be truncated by $\underline{z}$ or $\overline{z}$. If the agent is optimistic in the second stage, her optimization problem in the second stage is similar to (23) with V^P and θ^P replaced by V^O and θ^O, respectively, and the infimum in u is replaced by supremum. In this case, the optimal portfolio weight and the worst case belief distortion are

$$\pi_t = \frac{\mu - r}{\sigma^2} \frac{1}{\gamma + \theta^O Z_1^{1-\eta}} \quad \text{and} \quad u_t = \frac{\mu - r}{\sigma} \frac{\theta^O Z_1^{1-\eta}}{\gamma + \theta^O Z_1^{1-\eta}}, \quad t \geq 1. \tag{26}$$

Recall that $\theta^O < 0$. To ensure the agent is still effectively risk averse, we need $\gamma + \theta^O \overline{z}^{1-\eta} > 0$ when $\eta < 1$, or $\gamma + \theta^O \underline{z}^{1-\eta} > 0$ when $\eta > 1$. Without this condition, the portfolio choice problem is ill-posed for the optimistic agent.

If the agent starts with a pessimistic belief in the first stage, but switches to the optimistic state at time 1, comparing (24) and (26) shows that her effective risk aversion shrinks to $\gamma + \theta^O Z_1^{1-\eta}$, which is less than her effective risk aversion $\gamma + \theta^P Z_1^{1-\eta}$ had she remained pessimistic in the second stage, because $\theta^O < 0 < \theta^P$. Therefore, after switching to the optimistic state, the agent invests more in the risky asset. In addition, switching from pessimism to optimism changes u from positive to negative.

If the agent starts with an optimistic belief in the first stage, i.e., $u_0 < 0$, and remains optimistic in the second stage, positive return shocks to $B_1^{\mathbb{B}}$ increases $Z_1^{1-\eta}$, when $\eta < 1$. Hence agent's effective risk aversion decreases (due to $\theta^O < 0$), the risky investment increases, the belief distortion u becomes more negative, and the agent becomes more optimistic. In sufficiently good market conditions, $Z_1^{1-\eta}$ reaches $\overline{z}^{1-\eta}$, the agent's effective risk aversion attains its minimal value $\gamma + \theta^O \overline{z}^{1-\eta}$, and the agent becomes extremely optimistic. In extremely bad market conditions, if the agent still remains optimistic, the optimal u is close to 0 when $Z_1^{1-\eta}$ reaches $\underline{z}^{1-\eta}$. However, recall that it is more likely that the agent has already switched to the pessimistic state as $Z_1^{1-\eta}$ approaches $\underline{z}^{1-\eta}$.

The full stage 1 problem for the agent starting with a pessimistic belief is

$$V_0^P = \sup_{\pi,c} \inf_u \mathbb{E}^{\mathbb{U}} \Big[\int_0^1 e^{-\delta t} \Big(\delta u(c_t) + \frac{1-\gamma}{2\theta^P} V_t^P Z_0^{\eta-1} |u_t|^2 \Big) dt + e^{-\delta} p_1^P V_1^P + e^{-\delta}(1-p_1^P) V_1^O \Big], \quad (27)$$

where $V_1^O = \frac{W^{1-\gamma}}{1-\gamma} e^{f_1^O(Z_1)}$ and f_1^O has a similar expression as f_1^P with θ^P replaced by θ^O in footnote 16. The intuition gained from this two-stage model survives in the fully dynamic model developed in the next subsection.

2.3 Dynamic Optimal Consumption and Portfolio Choice

We can now solve for optimal consumption and portfolio choice. As illustrated by the two-stage example, Z is the central state variable that determines the agent's optimization problem in (21). For ease of exposition, in the following, we take a monotone transformation

$$x_t = \log Z_t. \quad (28)$$

We call x the *market sentiment variable* and take it as the state variable for problem (21). It measures the discrepancy between the agent's belief and the baseline belief $\mathbb{B}$, and therefore also the strength of sentiment. In the optimistic state, because u^* is negative, positive return shocks to $B^{\mathbb{B}}$ increase x, the agent becomes more optimistic, and the best-case belief moves further away from $\mathbb{B}$ when $\eta < 1$. In the pessimistic state, due to u^* being positive, negative return shocks increase x, making the agent more pessimistic and the worst-case belief moves further away from $\mathbb{B}$ when $\eta < 1$.

We choose the function Ψ as in (22) such that it ensures the following homothetic decomposition of the optimal value functions

$$V_t^P = \frac{W_t^{1-\gamma}}{1-\gamma} e^{f^P(t,x_t)} \quad \text{and} \quad V_t^O = \frac{W_t^{1-\gamma}}{1-\gamma} e^{f^O(t,x_t)}. \quad (29)$$

We can now obtain the Hamilton-Jacobi-Bellman (HJB) equations satisfied by f^P and f^O using dynamic programming and summarize the agent's optimal investment and consumption strategies in the following Proposition.

Proposition 3. *Functions f^P and f^O satisfy a system of HJB equations specified in Proposition 6 in Appendix A. Denote $\Sigma = \sigma\sigma'$ and*

$$\gamma^{\mathit{eff},I}(t,x) = \gamma + \frac{(1-\gamma)(1+\partial_x f^I)^2}{\partial_{xx}^2 f^I + \partial_x f^I + (\partial_x f^I)^2 + \frac{1-\gamma}{\theta^I} e^{(\eta-1)x}}, \quad I \in \{O,P\}, (t,x) \in [0,T) \times (\underline{x}, \overline{x}), \quad (30)$$

where $\underline{x} = \log \underline{z}$ and $\overline{x} = \log \overline{z}$. Suppose that $\gamma^{\mathit{eff},I}(t,x) > 0$, for any $(t,x) \in [0,T) \times (\underline{x}, \overline{x})$ and

$I \in \{O, P\}$. *Then the agent's optimal beliefs and strategies are given by*

$$u^{I,*} = \frac{(1-\gamma)(1+\partial_x f^I)}{\partial^2_{xx} f^I + \partial_x f^I + (\partial_x f^I)^2 + \frac{1-\gamma}{\theta^I} e^{(\eta-1)x}} \sigma' \pi^{I,*}, \tag{31}$$

$$\pi^{I,*} = \frac{1}{\gamma}\Sigma^{-1}\Big(\mu - r - (1+\partial_x f^I)\sigma u^{I,*}\Big) = \frac{1}{\gamma^{\textit{eff},I}}\,\Sigma^{-1}(\mu - r), \tag{32}$$

$$\frac{c^{I,*}}{W} = \delta^{\psi} e^{-\frac{1}{\gamma} f^I}, \quad I \in \{O, P\}. \tag{33}$$

The optimal portfolio weight in (32) generalizes our finding for the two-stage example and highlights the impact of the Cressie-Read penalty. Without a preference for robustness, CRRA utility as well as Epstein and Zin (1989) preferences in combination with i.i.d. returns lead to the well-known myopic portfolio, namely $\pi^* = \frac{1}{\gamma}\Sigma^{-1}(\mu - r)$, see, e.g., Samuelson (1969) and Merton (1969). In the case of entropy, the main effect in the pessimistic state is to increase effective risk aversion, replacing γ by $\gamma + \theta^P$, see, e.g., Maenhout (2004).

In the case of Cressie-Read divergence, we obtain two novel effects. First, effective risk aversion $\gamma^{\text{eff},I}$ becomes belief- and state-dependent, driven by the endogenous sentiment state variable, in line with the discussion of the utility index process in Section 1.2 earlier. Note that effective risk aversion simplifies to γ when the agent has no preference for robustness ($\theta^I = 0$) and becomes $\gamma + \theta^I$ when $\eta = 1$, i.e. in the entropy case, since f^I is state-independent in this case. Second, the investor anticipates future changes in beliefs and the corresponding changes in perceived investment opportunities. This induces a Merton-type intertemporal hedging demand, which is captured by the term $-\frac{1}{\gamma}\Sigma^{-1}\partial_x f^I \sigma u^{I,*}$, added to the mean-variance optimal portfolio $\frac{1}{\gamma}\Sigma^{-1}(\mu - r - \sigma u^{I,*})$.

The condition in Proposition 3 that $\gamma^{\text{eff},I} > 0$, i.e., the agent is still effectively risk-averse after taking into account the state-dependent component, gives rise to constraints on the choice of θ^O. However, θ^P can be an arbitrary positive constant.[17]

We now study the optimal portfolio weight in more detail numerically. To this end, we focus on a single risky asset ($d = 1$) and numerically solve the HJB equations in Proposition 3 with the parameters listed in Panel A of Table 2. We first shut down regime switching (i.e., $\lambda^O = \lambda^P = 0$) and examine portfolio choice under optimism and pessimism separately. Afterwards we discuss the additional effects due to regime switching.

We start by considering a short horizon T of one year in order to focus on the effect of the Cressie-Read penalty on the myopic portfolio component. In the limit when the investor's horizon shrinks to zero, the intertemporal hedging component vanishes and only the myopic component remains. When the sentiment state variable x is zero, we expect a portfolio

[17] When $\gamma > 1$, the second-order condition for the optimization in u implies that $\gamma^{\text{eff},P}$ is always positive (see condition (ii) in Proposition 6 in Appendix A). However, when $I = O$, the second-order condition implies that the second term on the right-hand side of (30) is negative (see condition (ii') in Proposition 6). This term cannot be too negative to dominate γ, lest it results in a negative effective risk aversion $\gamma^{\text{eff},O}$. We verify the conditions of Proposition 6 numerically in our examples later on.

Table 2. **Parameter Values**

This table reports parameter values used for simulations. In addition to the variables defined below we use $\epsilon = 1$.

Parameter	**Variable**	**Value**
	Panel A: Partial Equilibrium	
r	Interest Rate	0.03
δ	Discount Rate	0.03
μ	Expected Stock Return	0.1
σ	Stock Volatility	0.2
γ	Risk Aversion	6
	Panel B: General Equilibrium	
δ	Discount Rate	0.04
μ^c	Consumption Growth Rate	0.0191
σ^c	Consumption Volatility	0.038
μ^D	Dividend Growth Rate	0.0245
σ^D	Dividend Volatility	0.17
ρ	Consumption-Dividend Correlation	0.2
γ	Risk Aversion	7
ψ	EIS	1.3
η	Cressie-Read Parameter	0.6
θ^P	Preference parameter for pessimism	6
θ^O	Preference parameter for optimism	-1
ℓ^O	Intensity parameter from P to O	0.005
ℓ^P	Intensity parameter from O to P	0.025

allocation close to the entropy case. Figure 2 shows that the Cressie-Read penalty produces the same optimal distortion u^* and the same portfolio weight π^*, for any value of η, both in the optimistic and pessimistic states when x is zero. This coincides with the results obtained in the second stage of the two-stage example with $Z = 1$ in equations (24) and (26).

In the top two panels of Figure 2, the investor is pessimistic with robustness parameter $\theta^P = 1$. As discussed at the beginning of Section 2.3, the sentiment variable x increases when market conditions deteriorate. At the same time, when $\eta < 1$, effective risk aversion is countercyclical and increases with x, the agent becomes more pessimistic as u increases (top left panel) and portfolio choice becomes more conservative (top right panel). Moreover, risk aversion increases faster for smaller η, which explains the steeper slope for $\eta = 0.5$ than for $\eta = 0.7$. On the other hand, when $\eta > 1$, effective risk aversion decreases with x, as is apparent from the positively sloped portfolio rule for $\eta = 1.3$ and 1.5 in the top right panel.

Turning to optimism with parameter $\theta^O = -1$ in the bottom two panels of Figure 2, the sentiment variable x now increases as market conditions improve. For $\eta < 1$, effective risk aversion and belief distortion (bottom left panel) are countercyclical, and the optimal portfolio weight is procyclical (bottom right panel). The effects for $\eta > 1$ are opposite.

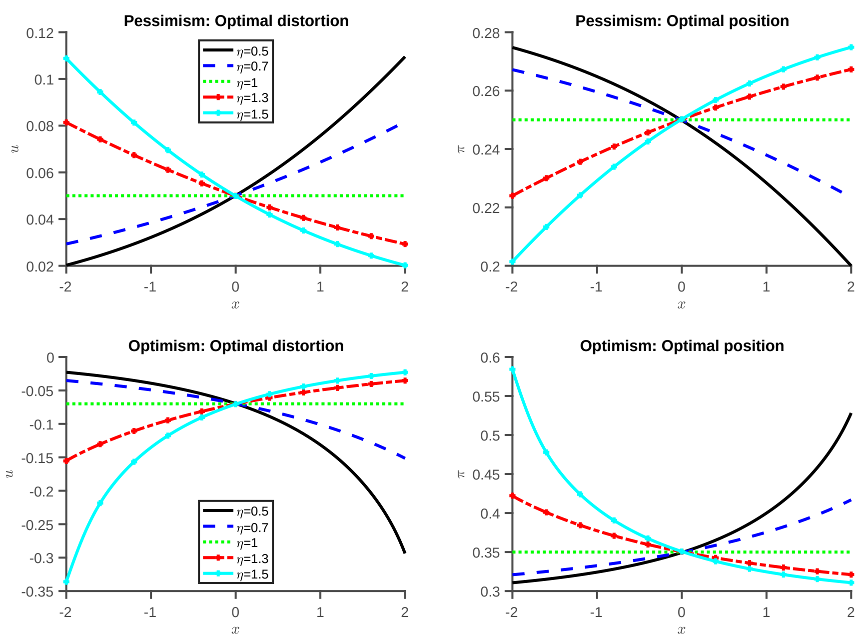


Figure 2. **State dependent optimal distortions and portfolios**

Notes: This figure plots optimal distortions and portfolios for pessimistic utility (top panels) and optimistic utility (bottom panels) at time zero. The preference parameters for robustness are $\theta^P = 1$ (top panels) and $\theta^O = -1$ (bottom panels). Other parameters used are summarized in Panel A of Table 2 and the time horizon is $T = 1$ year.

We now turn to a long horizon with T up to 100 years in order to study intertemporal hedging and resulting horizon effects. The seminal work of Merton (1973) explains how non-myopic investors tilt their portfolio in order to hedge against future changes in the investment opportunity set. For example, if returns on a risky asset are contemporaneously negatively correlated with expected returns on that asset, it becomes less risky to hold over longer horizons, inducing investors with longer horizons to increase their holdings. Despite returns being i.i.d. in our setting, intertemporal hedging appears because of the investor's distorted beliefs; even though expected returns are constant under the reference measure $\mathbb{B}$, they are not constant under $\mathbb{U}$ when $\eta \neq 1$. In particular, because of time-varying sentiment, the belief distortion u reacts to return innovations. When the belief distortion is countercyclical ($\eta < 1$), positive return shocks reduce pessimism or strengthen optimism, which increase the perceived expected return on the risky asset. In other words, for $\eta < 1$, returns and expected returns under $\mathbb{U}$ are positively correlated, which leads to negative intertemporal hedging demands that grow with the horizon, see the right two panels of Figure 3. Recall that for $\eta > 1$, positive return shocks strengthen pessimism or reduce optimism and therefore decrease the perceived expected return, we now have a negative correlation between historic returns and expected return. This explains the positive intertemporal hedging demand for $\eta > 1$, which naturally grows with the horizon, see the right two panels of Figure 3. We also observe from Figure 3 that the relationship between η and the sentiment state is asymmetric. When the agent is pessimistic, effects are more pronounced for $\eta < 1$ than for $\eta > 1$ for the same deviation from $\eta = 1$, for example, $\eta = 0.5$ and $\eta = 1.5$. Meanwhile, with optimism the effects are more pronounced for $\eta > 1$.

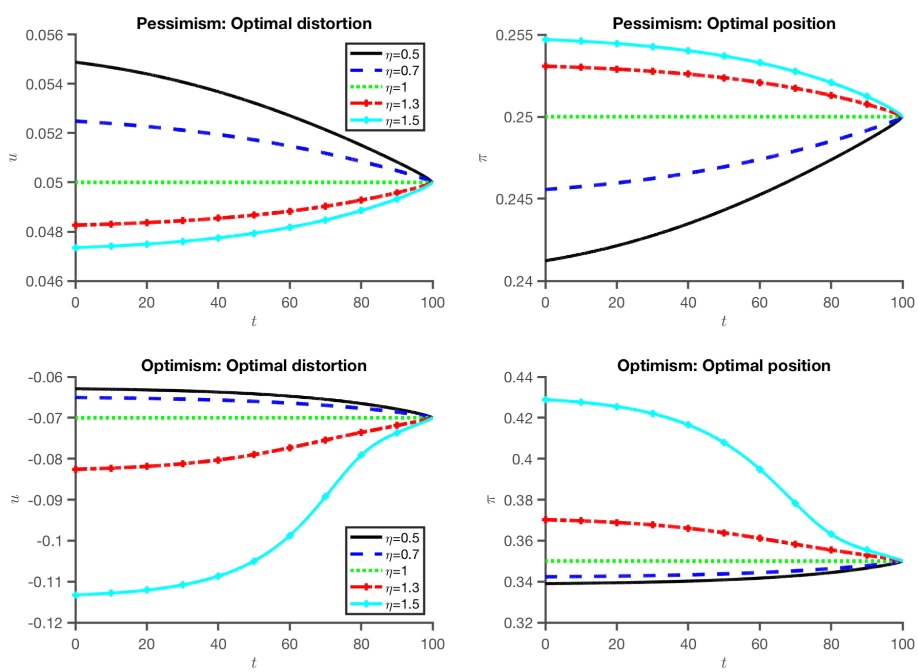


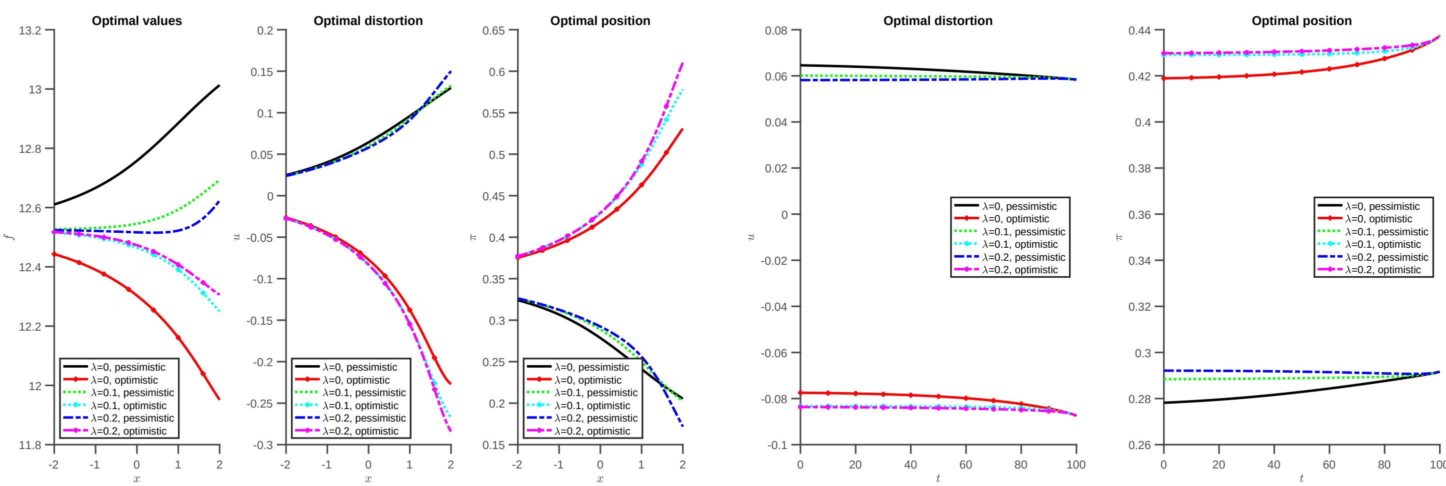
Figure 3. **Time dependence of optimal distortions and portfolios**

Notes: This figure plots optimal distortions and portfolios for pessimistic utility (top panels) and optimistic utility (bottom panels) at $x = 0$. The preference parameters for robustness are $\theta^P = 1$ (top panels) and $\theta^O = -1$ (bottom panels). Other parameters used are summarized in Panel A of Table 2 and the time horizon is $T = 100$ years.

We now discuss additional effects due to regime switching. To this end, we fix the regime-switching intensities in equation (19) to

$$\lambda^I = \ell^I / Z_t^{1-\eta} = \ell^I e^{(\eta-1)x_t}, \quad I \in \{I, O\}, \tag{34}$$

for positive constants ℓ^I. As ℓ^I increases, regime switching is more likely. As a result, the left panel of Figure 4 (a) shows that the agent's value functions f^P and f^O in (29) become more similar to each other as ℓ^I increases, because the continuation utility after regime switching is the optimal value of the other state. The middle and right panels in (a) demonstrate increasing state-dependence in the regime-switching model. The intuition for this is that after positive return shocks, the pessimistic agent not only becomes less pessimistic, but also more likely to switch to optimism. Similarly, after bad return shocks, the optimistic agent becomes less optimistic and more likely to become pessimistic. Therefore, regime switching strengthens the effect of dynamic optimism and pessimism, increasing the curvature of the portfolio allocation rules in the right panel of (a). The right panel of Figure 4 (b) shows that intertemporal hedging decreases with the regime-switching intensities. This is because large intensities effectively shrink the investor's horizon, which dampens the intertemporal hedging demand.



(a) State dependence at time zero

(b) Time dependence

Figure 4. **Value functions, optimal distortions and portfolios in regime switching models**

Notes: This figure plots value functions (f^P and f^O in (29)), optimal distortions, and portfolios in the regime switching model at time zero (left panels), and the time dependence of optimal distortions and portfolios at $x = 0$ (right panels). The preference parameters for robustness are $\theta^P = 1$ and $\theta^O = -1$. The transition intensities are $\lambda^P = \lambda^O = \lambda / \exp((1-\eta)x)$. Other parameters used are summarized in Panel A of Table 2, the Cressie-Read parameter is $\eta = 0.5$, and the time horizon is $T = 100$ years.

3 General Equilibrium

We now extend our analysis to equilibrium asset pricing. To this end, we consider an economy with a single Lucas tree. Its dividend, which must be consumed by the representative agent, has the following dynamics

$$\frac{dc_t}{c_t} = \mu^c dt + \sigma^c dB_t^{\mathbb{B}}, \tag{35}$$

where μ^c and σ^c are constants. The equilibrium price P^L of this Lucas tree follows

$$\frac{dP_t^L + c_t dt}{P_t} = \mu_t^L dt + \sigma_t^L dB_t^{\mathbb{B}}. \tag{36}$$

The expected return μ^L, the volatility σ^L, and the interest rate r will be determined endogenously. We take x as the state variable and conjecture that μ^L, σ^L, and r are all functions of time and the state.[18]

Given P^L and r, the representative agent, with preferences based on Cressie-Read divergence, invests and consumes with the optimal investment strategy π^* and the optimal consumption strategy c^*. In an attempt to match empirical asset pricing evidence quantitatively, we extend our earlier analysis and endow the representative agent with Epstein and Zin (1989) preferences towards intertemporal consumption in addition to Cressie-Read divergence.[19] We denote the agent's elasticity of intertemporal substitution by $0 < \psi \neq 1$ and define $\nu = \frac{1-\gamma}{1-\frac{1}{\psi}}$.

Definition 1. $(r, \mu^L, \sigma^L, \pi^*, c^*)$ is an equilibrium if

1. The financial market clears, i.e., $\pi^* \equiv 1$;
2. the aggregate resource constraint holds, i.e., $c^* \equiv c$.

We now present equilibrium quantities in the following Proposition.

Proposition 4. *Let f^P and f^O be solutions to a system of equations specified in Proposition 7 in Appendix A. Then the equilibrium expected return, volatility, and risk-free interest rate under*

[18]While the optimal portfolio and consumption decisions obtained in Section 2 under the assumption of constant μ, σ and r, the solution to the dynamic consumption and portfolio problem in Proposition 6 takes the state variable x into account. Therefore, even though μ^L, σ^L, and r are x-dependent, no additional state variable needs to be introduced and the optimal u^* and π^* still take the form of (31) and (32) with μ and σ therein replaced by μ^L and σ^L.

[19]The details of the preferences and the associated optimal consumption-investment problem are summarized in Appendix A.

regime $I \in \{O, P\}$ are given by

$$\mu^{L,I} - r^I = \gamma^{\textit{eff},I}(\sigma^{L,I})^2, \tag{37}$$

$$\sigma^{L,I} = \sigma^c - \frac{\psi}{\nu}\partial_x f^I u^{I,*}, \tag{38}$$

$$r^I = \delta + \frac{1}{\psi}\mu^c - \frac{1}{2}\gamma^{\textit{ent},I}\big(1 + \frac{1}{\psi}\big)(\sigma^c)^2 + DA^I, \tag{39}$$

where $u^{I,*}$ is given by

$$u^{I,*} = \frac{(1-\gamma)(1+\partial_x f^I)\sigma^c}{\partial^2_{xx} f^I + \psi\partial_x f^I + \psi(\partial_x f^I)^2 + \frac{1-\gamma}{\theta^I}e^{(\eta-1)x}}, \tag{40}$$

the effective risk aversion $\gamma^{\textit{eff},I}$ is given in (30) and $\gamma^{\textit{ent},I} = \gamma + \theta^I e^{(1-\eta)x}$ is the effective risk aversion in the entropy case with the robustness parameter frozen at $\theta^I e^{(1-\eta)x}$. DA^I is a state-dependent dynamic adjustment, whose expression is given in (A.14).

The equilibrium equity premium in (37) is given by a Consumption CAPM relationship, where the key innovation is the time-varying price of risk. The Cressie-Read penalty allows us to produce rich dynamics, despite the stylized underlying dynamics of this lognormal i.i.d. economy. Recall that we expect a countercyclical price of risk for $\eta < 1$, since in this case the investor's effective risk aversion $\gamma^{\text{eff},I}$ is countercyclical, due to increases in pessimism or decreases in optimism following adverse shocks. This mechanism is driven entirely by time-varying sentiment and the stochastic beliefs it generates endogenously. Juxtaposition with the entropy case reveals that even in the regime-switching model, equilibrium quantities remain constant in each sentiment state, because f^I is state-independent.[20] When the agent's sentiment state changes due to market-independent shocks in the regime-switching model, equilibrium quantities switch as well. Moreover, when $\eta < 1$, switching between different sentiment states is more likely when the sentiment variable x is low, i.e., when the agent's subjective belief is close to the reference model $\mathbb{B}$.[21]

The second key contribution to equilibrium asset pricing is that we obtain excess volatility, driven by time-varying sentiment. With entropy, $\sigma^{L,I} = \sigma^c$ and the well-known excess volatility puzzle emerges, as is common in standard asset pricing models with lognormal dynamics. To understand the intuition, we turn off regime switching (i.e. $\lambda^I \equiv 0$). In equation (38), observe that from equation (29)

$$-\frac{\psi}{\nu}\partial_x f^I u^{I,*} = \frac{\psi-1}{W^{1-\gamma}e^{f^I}}\Big(\partial_x V^I(-u^{*,I})\Big),$$

where V^I is the optimal value function. Note that $\partial_x V^I(-u^{*,I})$ corresponds to the sensitivity

[20]In the entropy cost case, f^I satisfies a pair of coupled ODEs obtained by setting $\eta = 1$ and all spatial derivatives to zero in (A.10). From this coupled ODE, we observe that f^I is time-dependent, but state-independent.

[21]This is because Λ^O and Λ^P are both decreasing in x when $\eta < 1$. Therefore the regime-switching intensities λ^O and λ^P are decreasing in x.

of the optimal value with respect to fundamental shocks, which is positive for a procyclical value function. Therefore, excess volatility emerges when the EIS ψ is in excess of 1.[22] It is also intuitive that the excess volatility generated in our model is proportional to $u^{I,*}$, since this is the instantaneous volatility of the (log) state variable Z capturing belief distortions. In particular, when the agent is pessimistic, $u^{*,P}$ increases in bad times, therefore we expect equilibrium stock price volatility to increase in bad times when the agent is more likely to be pessimistic. We confirm this in the calibrated model in the next section.

Finally, for the equilibrium risk-free rate, we present a decomposition to facilitate comparison with existing results in the literature so as to flesh out our contributions most clearly. We find that Cressie-Read penalty adds a rich dynamic adjustment to the equilibrium risk-free rate that obtains in an economy with entropy penalty. The first three terms in equation (39) represent the effect of the usual determinants of savings behavior on equilibrium interest rates, namely the rate of time preference δ, intertemporal substitution based on the investor's EIS ψ and expected consumption growth μ^c, and precautionary savings. The Cressie-Read setting adds a dynamic adjustment to these standard determinants. These additional terms are all related to precautionary savings reflecting the stochastic effective risk aversion and the higher volatility in the economy due to time-varying sentiment. For the entropy case with robustness preference parameter set to $\theta^I e^{(1-\eta)x}$, DA^I vanishes, reducing (39) exactly to the case in Maenhout (2004). Signing the DA^I requires solving the HJB (A.10), which we do in the calibration later.

Before turning to the calibration, we show how to use these equilibrium results to price a stock. To this end, consider dividend dynamics given by

$$\frac{dD_t}{D_t} = \mu^D dt + \sigma^D(\rho dB_t^{\mathbb{B}} + \sqrt{1-\rho^2}dB_t^{\perp}), \tag{41}$$

where μ^D and σ^D are constants representing the dividend growth rate and volatility respectively, and $B^{\perp}$ is a Brownian motion independent of $B^{\mathbb{B}}$. The constant ρ is the instantaneous correlation between consumption and dividend growth.

We consider the stock as an asset in zero net supply with a shadow price determined in equilibrium. Suppose that S follows the dynamics

$$\frac{dS_t + D_t dt}{S_t} = \mu_t^{S,I} dt + \sigma_t^{S,I} dB_t^{\mathbb{B}} + \sigma_t^{S,I,\perp} dB_t^{\perp}, \tag{42}$$

where $I \in \{O, P\}$. Define $\ell = S/D$ as the price-dividend ratio. The following result presents the equilibrium stock return and volatility in different sentiment states.

Proposition 5. *Let ℓ^I, $I \in \{O, P\}$, be the solution to the equation* (B.23) *in Online Appendix B*

[22] The necessity of Epstein and Zin (1989) preferences is also highlighted in Jin and Sui (2019) who study a model of asset pricing with extrapolative expectations. These preference parameter restrictions are also common in the long-run-risk literature.

and $u^{I,*}$ *given by* (40). *Then*

$$\begin{aligned} \mu_t^{S,I} =& \frac{\partial_t \ell^I}{\ell^I} + \frac{1}{2}|u^{I,*}|^2 \frac{\partial^2_{xx}\ell^I}{\ell^I} - \frac{1}{2}|u^{I,*}|^2 \frac{\partial_x \ell^I}{\ell^I} + \mu^D - u^{I,*}\sigma^D \rho \frac{\partial_x \ell^I}{\ell^I} + \frac{1}{\ell^I}, \\ \sigma_t^{S,I} =& - u^{I,*}\frac{\partial_x \ell^I}{\ell^I} + \sigma^D \rho, \qquad \sigma_t^{S,I,\perp} = \sigma^D \sqrt{1-\rho^2}. \end{aligned} \tag{43}$$

Moreover, the CAPM relation $\mu_t^{S,I} = r_t^I + \lambda_t^I \sigma^{S,I}$ *is also satisfied.*

The results are intuitive and extend our earlier findings to the case of an asset that pays dividends that are less than perfectly correlated with the consumption stream of the representative agent. The risk premium on the stock is given by the standard Consumption CAPM, but with a time-varying price of risk generated by our model. We also obtain excess volatility. Without robustness consideration or with entropy-based robustness, the price-dividend ratio is trivially constant in a lognormal economy, resulting in the equilibrium stock volatility being equal to the dividend volatility. The Cressie-Read divergence measure leads to time-varying beliefs, inducing a dynamic price-dividend ratio. Equation (43) shows that excess volatility emerges when the price-dividend ratio is procyclical. This is intuitive and reflects the contribution of volatile valuation ratios to stock return volatility. High price-dividend ratios driven by positive sentiment during good times, as well as low price-dividend ratios driven by negative sentiment in bad times both act to raise equilibrium stock volatility above dividend volatility. We examine the equilibrium stock volatility in our model calibration later.

4 Estimating Sentiment

We now estimate a measure of time-varying sentiment from the data. We first measure the difference between agents' subjective and objective beliefs. To measure subjective beliefs, we make use of an extensive survey on aggregate GDP growth, unemployment, and inflation. In addition to subjective beliefs, we also need a measure of objective beliefs since the difference between the subjective and objective beliefs will help us back out the optimal distortion. Objective beliefs are calculated from a vector autoregression (VAR) from which we infer forecasts of macroeconomic variables.

Subjective Beliefs: We hand collect survey data from Consensus Economics Inc. Each month survey participants are asked for their forecasts of a range of macroeconomic and financial variables for the major economies. There are on average around 30 respondents each month. Our analysis focuses on US real GDP growth, unemployment, and inflation.[23] Each month respondents submit a forecast for the current end of calendar year, as well as next end of

[23] We also use an alternative survey of macro forecasts, namely Blue Chip Economic Indicators, and find qualitatively similar results.

calendar year. Since these forecasts are formed over a moving forecast horizon, we use a linear interpolation method to get constant maturity forecasts.

Objective Beliefs: To get a proxy for objective beliefs, we estimate a VAR with two lags on real GDP, inflation, and unemployment and use forecasts from this VAR(2).

We define a wedge on beliefs as the difference between the subjective and objective beliefs about future macroeconomic variables, measuring the amount of pessimism or optimism in the economy. Figure 1 plots the wedge for all three variables using data between 1995 and 2018. The figure reveals that before economic recessions, beliefs are optimistic, whereas there is a dramatic drop in GDP wedges turning to pessimism after recessions. While the rebound after the 2001 crisis has been fast, the repercussions of the 2008 Great Financial Crisis are long lasting leading to significant pessimism on average. In our sample period, the average GDP wedge is around -1.06% with an associated standard deviation of 1.4%. Skewness is also negative at -0.904.

In the following, we use our GDP estimates to inform us about the dynamics of the optimal belief distortion $u^{\star}$. Recall that in our model the wedge on beliefs is defined as $-u^{*}\sigma^{L,I}$. Belief distortions $u^{\star}$ are therefore estimated by dividing the wedge by the sample standard deviation of the Lucas tree.

5 Calibration

We now turn to the calibration in order to explore the ability of our equilibrium model to quantitatively match salient features of asset prices together with the wedge dynamics. To this end, we assume that the representive agent's sentiment toward consumption is the same as the sentiment estimated via GDP growth in the previous section. We numerically solve the equations in Propositions 4 and 5. Importantly, we discipline the free parameters governing the preference for robustness, Cressie-Read divergence, and intensities of regime switching by the first three moments of the wedges, their sluggishness (measured by the AR(1) coefficient of wedges), and the proportion of time in different sentiment states reported in Panel A of Table 3. We use the values of consumption growth rate, consumption volatility, and dividend growth rate reported in Campbell and Cochrane (1999). The remaining model parameters used for calibration are summarized in Panel B of Table 2. In particular, the Cressie-Read parameter is crucial and our value $\eta = 0.6$ lies between entropy ($\eta = 1$) and Hellinger ($\eta = 0.5$). The preference parameters for pessimism and optimism are $\theta^{P} = 6$ and $\theta^{O} = -1$, respectively. Moreover, the regime-switching intensities take the form given in equation (34) with the intensity jumping from pessimism to optimism smaller than the other intensity, in order to match the wedge properties where 85% of the time the wedge is negative and the representative agent is pessimistic.

Table 3. **Summary Statistics of Equilibrium Quantities**

This table reports moments about equilibrium quantities of the calibrated model. The model is disciplined by the first three moments of wedges on GDP, its AR(1) coefficient, and the proportion of each sentiment state. The AR(1) coefficient is the fitted β in $\text{wedge}_t = \alpha + \beta \times \text{wedge}_{t-1} + \epsilon_t$, where $\text{wedge}_t = -u_t^* \sigma_t^L$. The empirical values of equilibrium quantities are obtained from Campbell and Cochrane (1999) and Beeler and Campbell (2012). The theoretical values are moments of equilibrium quantities between year 20 to 50 obtained by Monte Carlo simulation with 10^4 paths. Parameters used are summarized in Panel B of Table 2. The time horizon is $T = 100$ years.

Panel A: Statistic	**Calibrated value**	**Empirical value**
Mean wedge ($\mathbb{E}^{\mathbb{B}}[-u^* \sigma^L]$)	-0.97%	-1.06%
stdev wedge ($\sigma(-u^* \sigma^L)$)	0.84%	1.4%
Skewness wedge ($\text{skewness}(-u^* \sigma^L)$)	-1.48	-0.90
$AR(1)$ **coefficient on wedge**	0.92	0.83
Percentage of pessimistic periods	84%	85%
Panel B: Moments of equilibrium quantities	**Theoretical value**	**Empirical value**
Equity premium ($\mathbb{E}^{\mathbb{B}}[\mu^S - r]$)	4.29 %	3.90%
Stock volatility (σ^S)	17.7 %	18.0%
Sharpe ratio ($\mathbb{E}^{\mathbb{B}}[\mu^S - r]/\sigma^S$)	0.24	0.22
Interest rate ($\mathbb{E}^{\mathbb{B}}[r]$)	2.85 %	2.92%
Interest rate volatility ($\sigma(r)$)	1.42 %	2.89%
Mean log price-dividend ratio ($\mathbb{E}^{\mathbb{B}}(\log(P/D))$)	3.13	3.05
stdev log price-dividend ratio ($\sigma(\log(P/D))$)	0.10	0.27
Panel C: Conditional moments	**Pessimism**	**Optimism**
Equity premium ($\mathbb{E}^{\mathbb{B}}[\mu^S - r]$)	4.83 %	1.43%
Stock volatility (σ^S)	17.9 %	17.0%
Sharpe ratio ($\mathbb{E}^{\mathbb{B}}[\mu^S - r]/\sigma^S$)	0.27	0.08
Interest rate ($\mathbb{E}^{\mathbb{B}}[r]$)	2.45 %	4.88%
Interest rate volatility ($\sigma(r)$)	1.21 %	0.11%
Mean log price-dividend ratio ($\mathbb{E}^{\mathbb{B}}(\log(P/D))$)	3.12	3.19
stdev log price-dividend ratio ($\sigma(\log(P/D))$)	0.11	0.05

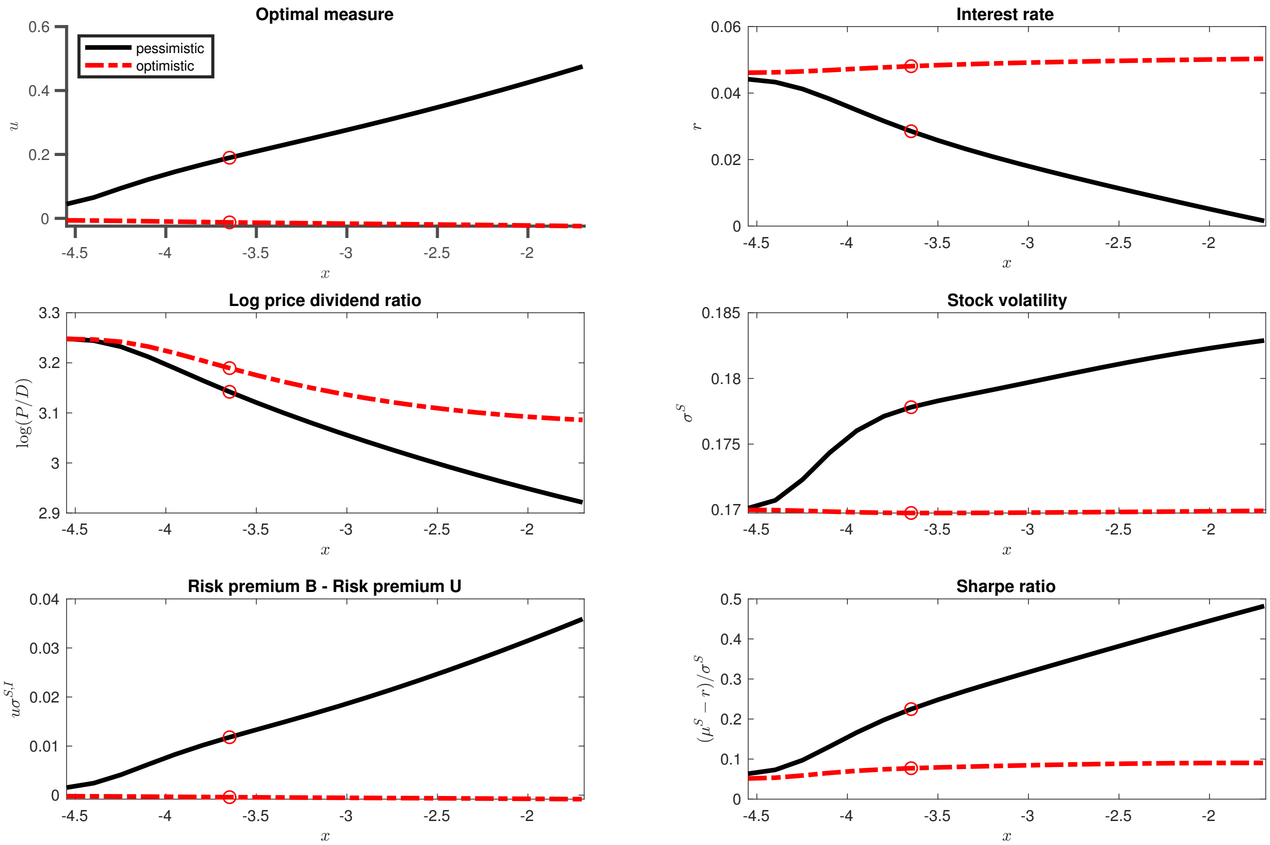


Figure 5. **General equilibrium quantities**

Notes: This figure plots the optimal distortion, interest rate, price-dividend ratio, stock volatility, difference in stock risk premia between measures, and the Sharpe ratio in equilibrium for both optimistic state (red dash lines) and pessimistic state (black solid lines). The stock volatility is $\sigma^S = \sqrt{(\sigma^{S,I})^2 + (\sigma^{S,I,\perp})^2}$. Parameters used are summarized in Panel B of Table 2 and time horizon is $T = 100$ years. All figures present quantities at 50 years. They all center at the mean of the sentiment variable and span the 5% to 95% quantiles of the distribution for the sentiment variable.

We follow Jin and Sui (2019) and use the empirical asset pricing evidence reported in Campbell and Cochrane (1999) and in Beeler and Campbell (2012) as objectives to match. Asset pricing results produced by the calibrated model are reported in Panel B of Table 3 and conditional moments of equilibrium quantities are reported in Panel C. Notice that most of our equilibrium results are driven by pessimism in our calibrated model because the data shows that wedges are mostly negative. In particular, Table 3 panel B shows that the model performs well in generating a sizeable risk premium and realistic Sharpe ratio. The quantitative success in producing excess volatility is more limited.

We can also study the effects of different states on equilibrium quantities. Figure 5 reports the state dependence at time $t = 50$ years. The black solid lines represent equilibrium quantities in the pessimistic state of the regime-switching model; while the red dashed lines represent equilibrium quantities in the optimistic state. Starting from the pessimistic state, successive positive fundamental shocks before year 50 move equilibrium quantities along the black solid lines to the left (i.e., smaller values of x). As x decreases, it is increasingly likely to switch to the optimistic state and jump to the red dashed line. Afterward continuing positive fundamental shocks drive equilibrium quantities along the red dashed lines to the right (i.e., larger value of x). Figure 5 shows that the belief distortion, equilibrium volatility, and Sharpe ratio are all countercyclical, the equilibrium interest rate is procyclical, the price-dividend ratio is procyclical in the pessimistic state and displays some reversal in the optimistic state, because the agent anticipates returning to the pessimistic state in the future. All of these results reflect the procyclical sentiment generated by the model.

Importantly, the wedge between the objective and the subjective risk premium is reasonable and around 130 basis points at the mean of the state space, while it spans approximately 360 basis points within the 5% to 95% quantiles of the state space. As Hansen and Sargent (2020) and Chamberlain (2020) both point out, a central idea in robust Bayesian analysis based on classical work of Good (1952) is to judge the plausibility of a min-max model by examining how reasonable the subjective belief $\mathbb{U}$ is that is supporting the equilibrium. We conclude that our deviations are therefore not too far off.

Turning to the distributions of the equilibrium quantities, Figure 6 shows heavy-tailed distributions in adverse market conditions for equilibrium interest rate, Sharpe ratio, and the equity risk premium. This echoes the elevated sentiment volatility after bad shocks in the pessimistic state discussed in Section 1.2. Even though sentiment volatility can also increase after good shocks in the optimistic state, the predominance of pessimistic sentiment in the data mutes the impact of optimistic periods.

Finally, we can use the same set of parameters to gauge the effect of Cressie-Read relative to entropy. For example, in the case of relative entropy, the model produces a higher risk-free rate of 3.9%, a very low equity risk premium of 1.5% percent, a constant return volatility equal to the volatility of dividends, and a constant Sharpe Ratio of 0.09. We therefore conclude by noting that our Cressie-Read extension improves substantially on the quantitative front, in addition to generating meaningful time-variation at the business cycle frequency.

6 Conclusions

Our paper makes the following contributions. First, we propose a new model of dynamic sentiment motivated and supported by empirical evidence that survey expectations deviate from rational expectations and exhibit prolonged episodes of pessimism and optimism. Forecast errors relative to rational expectations arise endogenously in our model as agents fear misspecification of the benchmark model and seek robustness by considering alternative models. We generalize existing approaches to robustness in two ways. First, we replace the entropy criterion that is ubiquitous in the literature on robust control by the

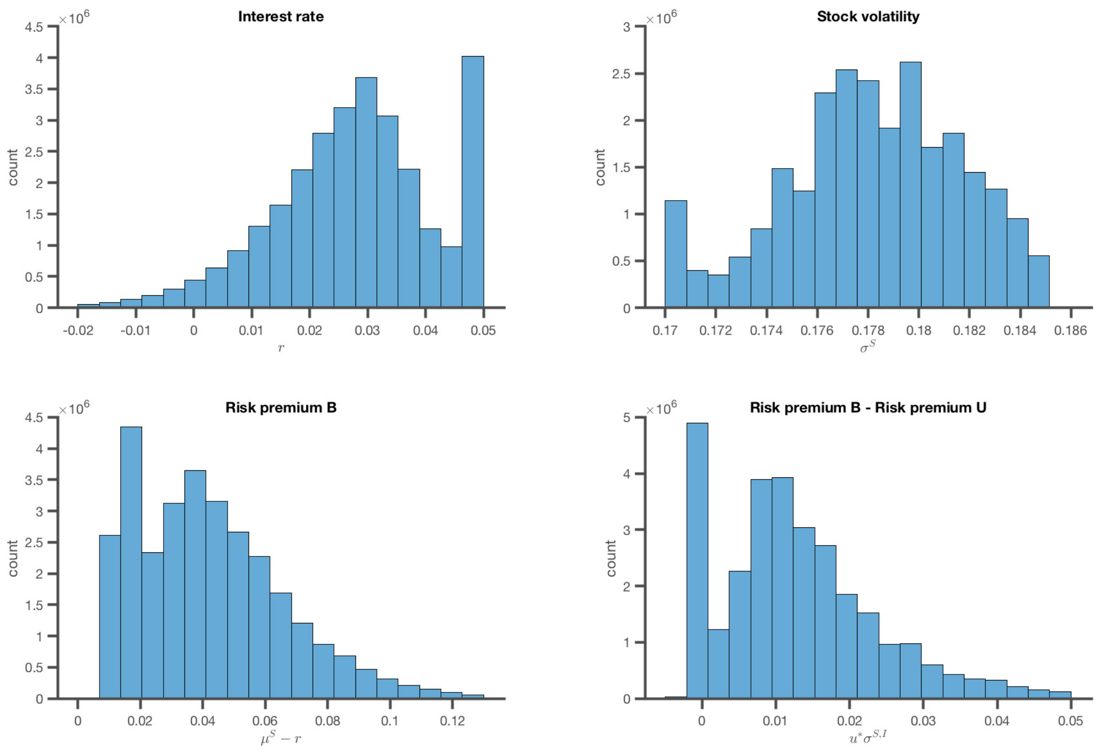


Figure 6. **Distribution of equilibrium quantities**

Notes: This figure plots the distributions of interest rate, stock volatility, risk premium $\mathbb{B}$, and difference in stock risk premia between measures along equilibrium path generated via Monte Carlo between year 20 and year 50. Parameters used are summarized in Panel B of Table 2 and time horizon is $T = 100$ years.

family of Cressie-Read divergences. Our Cressie-Read divergence measure is recursive and homothetic, leading to time-consistent preferences and tractable decision problems. This extension opens the door to endogenous and state-dependent dynamics of belief distortions, governed by the Cressie-Read parameter η. In particular, the dynamics of sentiment can be procyclical or countercyclical depending on the sign of $1 - \eta$. Sentiment is driven by past fundamental shocks, as well as by past belief distortions, thereby generating the sluggishness of sentiment observed empirically. As a second extension of the literature on robust control our model allows for optimism in addition to pessimism, and features a regime-switching mechanism where the likelihood of switching depends also on the sentiment state variable.

As a second contribution, we apply our model to understand portfolio choice and general equilibrium asset pricing. We highlight the ability of the model to generate rich dynamics and to explain empirically relevant phenomena by deliberately assuming i.i.d. Gaussian fundamentals, in which case entropy produces constant portfolios, risk premia, interest rates, and volatility. In our model, endogenous sentiment gives rise to stochastic effective risk aversion. When the Cressie-Read parameter η is smaller than one, effective risk aversion is countercyclical, while the opposite happens whenever $\eta > 1$. In our portfolio problem, this induces intertemporal hedging and therefore both horizon- and state-dependent portfolios, despite returns being i.i.d. We calibrate our general equilibrium asset pricing model using estimates of belief distortions obtained from survey data of expectations by professional forecasters about future economic activity. We find that our model is able to match empirical observations on equity premium, Sharpe ratio, and interest rates.

The model we proposed can be applied in a variety of dynamic decision problems that are of economic interest. For example, Ling, Miao, and Wang (2021) use our theoretical results to study robust financial contracting and corporate investment.

While we have deliberately studied the simplest possible Lucas economy to flesh out most clearly the implications of the Cressie-Read divergence, a natural extension of our work would feature tail risk. It is well-known that macroeconomic fundamentals such as consumption feature fat tails, which might be due to a small probability of a disaster, see, e.g., Barro (2006). Rare disaster asset pricing models study the implications of these large negative shocks for asset prices, see, e.g.,Tsai and Wachter (2015) for a review. A setting with tail risk and agents featuring robustness concerns is therefore a natural extension of our framework, which we leave for future research.

Giglio, Maggiori, Stroebel, and Utkus (2021) document rich heterogeneity in investor beliefs. Studying investor heterogeneity in beliefs and portfolios, including its effect on equilibrium asset prices is another very promising avenue for future research based on our model.

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Appendix A Additional results

Optimal consumption and investment for an Epstein-Zin utility agent

Consider an agent whose preference over consumption streams is described by a continuous-time stochastic differential utility of the Kreps-Porteus and Epstein-Zin type. Given a discount rate δ, relative risk aversion $0 < \gamma \neq 1$, and EIS $0 < \psi \neq 1$, the Epstein-Zin aggregator F (see, e.g., Duffie and Epstein (1992)) is $F(c, v) \equiv \delta \frac{c^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}} \big((1-\gamma)v\big)^{1-\frac{1}{\nu}} - \delta \nu v$, with $\nu = \frac{1-\gamma}{1-\frac{1}{\psi}}$. Incorporating the Cressie-Read divergence, we introduce the pessimistic Epstein-Zin preference for a consumption stream c as

$$\mathcal{U}_t^{P,c} = \inf_u \mathbb{E}_t^{\mathbb{U}} \Big[\int_t^T F(c_s, \mathcal{U}_s^{P,c}) + \tfrac{1}{2\theta^P} \Psi_s Z_{s\wedge\tau}^{\eta-1} |u_s|^2 \, ds + \epsilon U(c_T) \Big], \tag{A.1}$$

where $\tau = \inf\{t \geq 0 : Z_t \leq \underline{z} \text{ or } Z_t > \overline{z}\}$. Following Lemma 2, $\mathcal{U}^{P,c}$ satisfies the BSDE

$$d\mathcal{U}_t^{P,c} = \Big[\delta \mathcal{U}_t^{P,c} - F(c_t, \mathcal{U}_t^{P,c}) + \frac{\theta^P}{2\Psi_t} Z_{t\wedge\tau}^{1-\eta} |\Gamma_t|^2 \Big] dt + \Gamma_t' dB_t^{\mathbb{B}}, \quad \mathcal{U}_T^c = \epsilon U(c_T),$$

The worst-case belief distortion is induced by $u_t^* = \frac{\theta^P \Gamma_t}{\Psi_t} Z_{t\wedge\tau}^{1-\eta}$. The optimistic Epstein-Zin preference $\mathcal{U}^{O,c}$ and its associated best-case belief distortion can be characterized similarly.

In the regime switching model, consider the optimal consumption-investment problem (21) for an agent with Epstein-Zin preferences. V^P in (21) satisfies

$$V_t^P = \sup_{\pi,c} \inf_u \mathbb{E}_t^{\mathbb{U}} \Big[\int_t^{T\wedge\nu} F(c_s, V_s^P) + \frac{1-\gamma}{2\theta^P} V_s^P Z_{s\wedge\tau}^{\eta-1} |u_s|^2 \, ds + \epsilon U(c_T) \, 1_{\{\nu > T\}} + V_\nu^O 1_{\{\nu \leq T\}} \Big]. \tag{A.2}$$

$$V_t^O = \sup_{\pi,c} \sup_u \mathbb{E}_t^{\mathbb{U}} \Big[\int_t^{T\wedge\nu} F(c_s, V_s^O) + \frac{1-\gamma}{2\theta^O} V_s^O Z_{s\wedge\tau}^{\eta-1} |u_s|^2 \, ds + \epsilon U(c_T) \, 1_{\{\nu > T\}} + V_\nu^P 1_{\{\nu \leq T\}} \Big]. \tag{A.3}$$

We choose $x_t = \log Z_t$ as the agent's state variable. The choice of Ψ in (22) ensures the decomposition $V_t^P = \frac{W_t^{1-\gamma}}{1-\gamma} e^{f^P(t,x_t)}$ and $V_t^O = \frac{W_t^{1-\gamma}}{1-\gamma} e^{f^O(t,x_t)}$. The following proposition presents the equations satisfied by f^O, f^P and the optimal consumption and investment strategies.

Proposition 6. *When $\gamma \in (0,1)$, the function f^P defined in (29) satisfies*

$$\begin{aligned} 0 = \sup_{\pi,\tilde{c}} \inf_u \Big\{ & \partial_t f^P + \tfrac{1}{2}|u|^2 \Big(\partial_{xx}^2 f^P + \partial_x f^P + (\partial_x f^P)^2 \Big) - (1-\gamma)\partial_x f^P \pi' \sigma u + \delta \nu \tilde{c}^{1-\frac{1}{\psi}} e^{-\frac{1}{\nu} f^P} \\ & + (1-\gamma)\big[r + \pi'(\mu - r - \sigma u) - \tilde{c} - \tfrac{1}{2}\gamma \pi' \Sigma \pi \big] - \delta\nu + \tfrac{1-\gamma}{2\theta^P} e^{(\eta-1)x} |u|^2 + \Lambda^O(e^{(1-\eta)x}) \big(e^{f^O - f^P} - 1 \big) \Big\}, \end{aligned} \tag{A.4}$$

for $(t, x) \in [0, T) \times (\underline{x}, \overline{x})$, with boundary conditions

$$f^P(t, \underline{x}) = f_{\underline{x}}^{ent,P}(t), \quad f^P(t, \overline{x}) = f_{\overline{x}}^{ent,P}(t), \quad \textit{and} \quad f(T, x) = \log \epsilon. \tag{A.5}$$

When $\gamma > 1$, the infimum and supremum in (A.4) are changed to $\inf_{\pi,\tilde{c}} \sup_u$*. Denote*

$$\gamma^{\textit{eff},P}(t,x) = \gamma + \frac{(1-\gamma)(1+\partial_x f^P)^2}{\partial_{xx}^2 f^P + \partial_x f^P + (\partial_x f^P)^2 + \frac{1-\gamma}{\theta^P} e^{(\eta-1)x}}. \tag{A.6}$$

Suppose that, for any $(t,x) \in [0,T) \times (\underline{x}, \overline{x})$,

(i) $\gamma^{\textit{eff},P}(t,x)\Sigma$ *is positive definite,*

(ii) $(1-\gamma)\Big(\partial_{xx}^2 f^P + \partial_x f^P + (\partial_x f^P)^2 + \frac{1-\gamma}{\theta^P} e^{(\eta-1)x}\Big) > 0.$

Then the agent's optimal belief $u^{P,*}$ and strategies $\pi^{P,*}$ and $c^{P,*}$ are given by

$$\pi^{P,*} = \frac{1}{\gamma^{\textit{eff},P}} \Sigma^{-1}(\mu - r), \tag{A.7}$$

$$u^{P,*} = \frac{(1-\gamma)(1+\partial_x f^P)}{\partial_{xx}^2 f^P + \partial_x f^P + (\partial_x f^P)^2 + \frac{1-\gamma}{\theta^P} e^{(\eta-1)x}} \sigma' \pi^{P,*}, \tag{A.8}$$

$$\frac{c^{P,*}}{W} = \delta^\psi e^{-\frac{\psi}{\nu} f^P}. \tag{A.9}$$

Function $f_x^{\textit{ent},P}$ in (A.5) is the value for the problem with an entropy cost and $\theta(x) = \theta^P e^{(1-\eta)x}$. It then satisfies the ODE given in Online Appendix (B.12).

Equation (A.4) is coupled with f^O, which satisfies an equation similar to (A.4) with f^O and f^P swapped, θ^P replaced by θ^O, and $\sup_{\pi,\tilde{c}} \sup_u$ *when $\gamma \in (0,1)$ or* $\inf_{\pi,\tilde{c}} \inf_u$ *when $\gamma > 1$. The effective risk aversion $\gamma^{\textit{eff},O}$ is defined as (A.6) with f^P and θ^P replaced by f^O and θ^O, respectively. Conditions (i) and (ii) above are replaced by*

(i') $\gamma^{\textit{eff},O}(t,x)\Sigma$ *is positive definite,*

(ii') $(1-\gamma)\Big(\partial_{xx}^2 f^O + \partial_x f^O + (\partial_x f^O)^2 + \frac{1-\gamma}{\theta^O} e^{(\eta-1)x}\Big) < 0.$

Agent's optimal optimistic belief $u^{O,*}$ and strategies $\pi^{O,*}$ and $c^{O,*}$ are similar to (A.7), (A.8), (A.9) with f^P and θ^P replaced by f^O and θ^O.

Full statement of Proposition 4

Proposition 7. *Let f^I and $f^{\bar{I}}$ be solutions of the following coupled equations*

$$\begin{aligned} 0 = {} & \partial_t f^I + \tfrac{1}{2}|u^{I,*}|^2\Big(\partial_{xx} f^I + \partial_x f^I + \psi(\partial_x f^I)^2\Big) - (1-\gamma)\partial_x f^I u^{I,*}\sigma^c - \delta\tfrac{\nu}{\psi} \\ & + \tfrac{1-\gamma}{\psi}\Big(\mu^c - u^{I,*}\sigma^c - \tfrac{1}{2}\gamma(\sigma^c)^2\Big) + \delta^\psi \tfrac{\nu}{\psi} e^{-\frac{\psi}{\nu} f} + \tfrac{1-\gamma}{2\psi\theta} e^{(\eta-1)x}|u^{I,*}|^2 + \tfrac{1}{\psi}\Lambda^{\bar{I}}(e^{(1-\eta)x})\Big(e^{f^{\bar{I}} - f^I} - 1\Big), \end{aligned} \tag{A.10}$$

for $I \in \{O, P\}$, $\bar{I} = \{O,P\} \setminus I$, $(t,x) \in [0,T) \times (\underline{x}, \overline{x})$, with boundary conditions

$$f^I(t,\underline{x}) = F_{\underline{x}}^{\textit{ent},I}(t), \quad f^I(t,\overline{x}) = F_{\overline{x}}^{\textit{ent},I}(t), \quad f^I(T,x) = \log \epsilon. \tag{A.11}$$

Moreover u^* *in* (A.10) *is given by*

$$u^{I,*} = \frac{(1-\gamma)(1+\partial_x f^I)\sigma^c}{\partial_{xx}^2 f^I + \psi\partial_x f^I + \psi(\partial_x f^I)^2 + \frac{1-\gamma}{\theta}e^{(\eta-1)x}}. \tag{A.12}$$

In (A.11), $F_x^{ent,I}$, *with* $x = \underline{x}$ *or* $\overline{x}$, *is the value in an equilibrium where the representative agent has an entropy-based preference with the preference parameter* $\theta^I e^{(1-\eta)x}$ *and* $F_x^{ent,I}$ *satisfies coupled ODEs*

$$\begin{aligned} 0 =& \partial_t F_x^{ent,I} - \delta\frac{\nu}{\psi} + \frac{1-\gamma}{\psi}\big(\mu^c - u^{ent,I}\sigma^c - \tfrac{1}{2}\gamma(\sigma^c)^2\big) + \delta^\psi\frac{\nu}{\psi}e^{-\frac{\psi}{\nu}F_x^{ent,I}} \\ &+ \frac{1-\gamma}{2\psi\theta}e^{(\eta-1)x}|u^{ent,I}|^2 + \frac{1}{\psi}\Lambda^{\bar{I}}(e^{(1-\eta)x})\Big(e^{F^{ent,\bar{I}} - F^{ent,I}} - 1\Big), \end{aligned} \tag{A.13}$$

with the boundary condition $F_x^{ent,I}(T) = \log\epsilon$ *and* $u^{ent,I} = \theta^I e^{(1-\eta)x}\sigma^c$.

Then the equilibrium expected return, volatility, and risk-free interest rate are given in (37), (38), (39). *In* (39), *DA is given by*

$$\begin{aligned} DA^I =& -\gamma^{dyn,I}(\sigma^c)^2 - \frac{\psi\gamma}{\nu}\partial_x f^I u^{I,*}\sigma^c + (1-\tfrac{1}{\psi})u^{dyn,I}\sigma^c \\ &+ \big[-\tfrac{\psi}{\nu}\partial_x f^I + \tfrac{\psi^2}{2\nu^2}\big(1-\nu-2\gamma^{imp,I}\big)(\partial_x f^I)^2\big]|u^{I,*}|^2 - \frac{1-\frac{1}{\psi}}{2\theta^I}e^{(\eta-1)x}\big[2u^{ent,I}u^{dyn,I} + (u^{dyn,I})^2\big], \end{aligned} \tag{A.14}$$

where $\gamma^{eff,I} = \gamma^{ent,I} + \gamma^{dyn,I}$ *and* $u^{I,*} = u^{ent,I} + u^{dyn,I} = \theta^I e^{(1-\eta)x}\sigma^c + u^{dyn,I}$.

Online Appendix
–Not For Publication–

Appendix B Proofs

Continuous-time Cressie-Read divergence

Lemma 1. *When* $\Phi_t = Z_t^{1-\eta}$ *and* $\mathbb{E}^{\mathbb{B}}\big[\int_0^T e^{-\delta s}|\Psi_s|^p ds\big] < \infty$ *for some* $p > 2$*, then* $R^{\mathbb{U}}$ *in* (10) *becomes* (11)*, and it satisfies the following recursive relation*

$$R_t^{\mathbb{U}} = \mathbb{E}_t^{\mathbb{U}}\Big[\int_t^{\tilde{t}} e^{-\delta(s-t)}\frac{1}{2}\Psi_s Z_s^{\eta-1}|u_s|^2 ds + e^{-\delta(\tilde{t}-t)}R_{\tilde{t}}^{\mathbb{U}}\Big], \quad \textit{for any } \tilde{t} \geq t.$$

Proof. Using Itô's formula on $D_{t,s}$ defined in equation (10) yields that

$$dD_{t,s} = d\phi(Z_{t,s}) = \tfrac{Z_{t,s}-Z_{t,s}^{\eta}}{1-\eta}(-u_s')dB_s^{\mathbb{B}} + \tfrac{1}{2}Z_{t,s}^{\eta}|u_s|^2 ds. \tag{B.1}$$

It follows from Hölder's inequality that

$$\mathbb{E}_t^{\mathbb{B}}\Big[\int_t^T e^{-\delta(s-t)}(Z_{t,s}-Z_{t,s}^{\eta})^2\Psi_s^2|u_s|^2 ds\Big] \leq C^2\mathbb{E}_t^{\mathbb{B}}\Big[\int_t^T e^{-\delta(s-t)}(Z_{t,s}-Z_{t,s}^{\eta})^{2q}ds\Big]^{\frac{1}{q}}\mathbb{E}_t^{\mathbb{B}}\Big[\int_t^T e^{-\delta(s-t)}\Psi_s^{2p}ds\Big]^{\frac{1}{p}},$$

where $C = \max|u|$ and $1/p + 1/q = 1$. Because $\mathbb{E}_t^{\mathbb{B}}\big[\int_t^T e^{-\delta(s-t)}\Psi_s^{2p}ds\big] < \infty$ with some $p > 1$ by assumption and $\mathbb{E}_t^{\mathbb{B}}\Big[\int_t^T e^{-\delta(s-t)}(Z_{t,s} - Z_{t,s}^{\eta})^{2q}ds\Big] < \infty$ due to the boundedness of u, the process $\{e^{-\delta(s-t)}\Psi_s(Z_{t,s} - Z_{t,s}^{\eta})u_s\}_{s\geq t}$ is square integrable under $\mathbb{B}$. Hence $\int_t^{\cdot} e^{-\delta(s-t)}\Psi_s(Z_{t,s} - Z_{t,s}^{\eta})(-u_s')dB_s^{\mathbb{B}}$ is a martingale under $\mathbb{B}$. Then we have from (10) and (B.1) that

$$R_t^{\mathbb{U}} = \frac{1}{2\Phi_t}\mathbb{E}_t^{\mathbb{B}}\Big[\int_t^T e^{-\delta(s-t)}\Psi_s Z_{t,s}^{\eta}|u_s|^2 ds\Big].$$

When $\Phi_t = Z_t^{1-\eta}$,

$$R_t^{\mathbb{U}} = \frac{1}{2}\mathbb{E}_t^{\mathbb{B}}\Big[\int_t^T e^{-\delta(s-t)}\Psi_s Z_{t,s}Z_s^{\eta-1}|u_s|^2 ds\Big] = \frac{1}{2}\mathbb{E}_t^{\mathbb{U}}\Big[\int_t^T e^{-\delta(s-t)}\Psi_s Z_s^{\eta-1}|u_s|^2 ds\Big].$$

Then

$$\begin{aligned} R_t^{\mathbb{U}} =& \mathbb{E}_t^{\mathbb{U}}\Big[\int_t^{\tilde{t}} e^{-\delta(s-t)}\frac{1}{2}\Psi_s Z_s^{\eta-1}|u_s|^2 ds + e^{-\delta(\tilde{t}-t)}\int_{\tilde{t}}^T e^{-\delta(s-\tilde{t})}\frac{1}{2}\Psi_s Z_s^{\eta-1}|u_s|^2 ds\Big] \\ =& \mathbb{E}_t^{\mathbb{U}}\Big[\int_t^{\tilde{t}} e^{-\delta(s-t)}\frac{1}{2}\Psi_s Z_s^{\eta-1}|u_s|^2 ds + e^{-\delta(\tilde{t}-t)}R_{\tilde{t}}^{\mathbb{U}}\Big]. \end{aligned}$$

Non-Markovian utility index

For a given consumption stream c, which may not be Markovian, we use the theory of stochastic maximum principle (see, e.g., Bismut (1978)) to characterize the worst-case belief distortion in the following Lemma. We only present the result for the pessimistic utility in (13), the optimistic utility is similar with $\mathcal{U}^{P,c}$ and θ^P replaced by $\mathcal{U}^{O,c}$ and θ^O, respectively.

Lemma 2. *The worst-case belief distortion satisfies*

$$u_t^* = \frac{\theta^P \Gamma_t [1 + E_t]}{\Psi_t} Z_{t\wedge\tau}^{1-\eta}, \tag{B.2}$$

for some d-dimensional processes Γ and E. The pessimistic utility $\mathcal{U}^{P,c}$ follows the dynamics

$$d\mathcal{U}_t^{P,c} = \Big[\delta \mathcal{U}^{P,c} - \delta U(c_t) + \frac{\theta^P}{2\Psi_t} Z_{t\wedge\tau}^{1-\eta} |\Gamma_t|^2 \Big(1 - |E_t|^2\Big)\Big] dt + \Gamma_t' dB_t^{\mathbb{B}}, \quad \mathcal{U}_T^c = \epsilon U(c_T). \tag{B.3}$$

Proof of Lemma 2. Consider a fixed u and define its associated pessimistic utility index

$$\mathcal{U}_t^{P,c,u} = \mathbb{E}_t^{\mathbb{U}} \Big[\int_t^T e^{-\delta(s-t)} \delta U(c_s) ds + e^{-\delta(T-t)} \epsilon U(c_T) + \frac{1}{\theta^P} R_t^{\mathbb{U}} \Big].$$

The martingale representation theorem ensures the existence of a vector-valued process Γ^u such that

$$d\mathcal{U}_t^{P,c,u} = \big[\delta \mathcal{U}_t^{P,c,u} - \delta U(c_t)\big] dt - \Big\{ \tfrac{1}{2\theta^P} \Psi_t Z_{t\wedge\tau}^{\eta-1} |u_t|^2 - (\Gamma_t^u)' u_t \Big\} dt + (\Gamma_t^u)' dB_t^{\mathbb{B}}, \tag{B.4}$$

with the terminal condition $\mathcal{U}_T^{P,c,u} = \epsilon U(c_T)$. We define the pessimistic utility of c as

$$\mathcal{U}_t^{P,c} = \inf_u \mathcal{U}_t^{P,c,u}. \tag{B.5}$$

To identify the worst-case belief distortion u^*, we use the stochastic maximum principle (cf. Bismut (1978)). Introduce the Hamiltonian

$$H(Z, \mathcal{U}, \Gamma, \gamma, \widetilde{\mathcal{U}}, \widetilde{\Gamma}, u) = \gamma f(Z, \mathcal{U}, \Gamma, u) + \widetilde{\mathcal{U}} b(Z, u) + \widetilde{\Gamma}' \sigma(Z, u),$$

where $f(Z, \mathcal{U}, \Gamma, u) = -\delta \mathcal{U} + \delta U(c_t) + \big[\tfrac{1}{2\theta^P} \Psi_t Z_{t\wedge\tau}^{\eta-1} |u_t|^2 - \Gamma_t' u_t\big]$, $b(Z, u) \equiv 0$, and $\sigma(Z, u) = -Zu$ are the drift and volatility of the state variable Z, respectively. The adjoint variables γ and $\widetilde{\mathcal{U}}$ follow the dynamics

$$\begin{aligned} d\gamma_t &= \partial_{\mathcal{U}} H dt + \partial_\Gamma H dB_t^{\mathbb{B}}, \quad \gamma_0 = 1, \\ d\widetilde{\mathcal{U}}_t^u &= -\partial_Z H dt + \partial_\sigma H dB_t^{\mathbb{B}}, \quad \widetilde{\mathcal{U}}_T^u = 0, \end{aligned}$$

where $\partial_{\mathcal{U}} H$, $\partial_\Gamma H$, $\partial_Z H$, $\partial_\sigma H$ are partial derivatives with respect of H. Calculation shows that

$\gamma_t = e^{-\delta t} Z_t$ and

$$d\widetilde{\mathcal{U}}_t^u = -\Big[e^{-\delta t} \tfrac{\eta-1}{2\theta^P} \Psi_t Z_{t\wedge\tau}^{\eta-1} |u_t|^2 - \widetilde{\Gamma}_t^u u_t\Big] dt + \big(\widetilde{\Gamma}_t^u\big)' dB_t^{\mathbb{B}}, \quad \widetilde{\mathcal{U}}_T^u = 0.$$

Introduce $\overline{\mathcal{U}}_t^u = e^{\delta t} \widetilde{\mathcal{U}}_t^u$ and $\overline{\Gamma}_t^u = e^{\delta t} \widetilde{\Gamma}_t^u$. $\overline{\mathcal{U}}^u$ follows the dynamics

$$d\overline{\mathcal{U}}_t^u = \delta \overline{\mathcal{U}}_t^u dt - \Big[\tfrac{\eta-1}{2\theta^P} \Psi_t Z_{t\wedge\tau}^{\eta-1} |u_t|^2 - \overline{\Gamma}_t^u u_t\Big] dt + \big(\overline{\Gamma}_t^u\big)' dB_t^{\mathbb{B}}, \quad \overline{\mathcal{U}}_T^u = 0. \tag{B.6}$$

We can interpret $\overline{\mathcal{U}}_t^u$ as the marginal utility or shadow price with respect to the state variable Z.

The stochastic maximum principle implies that the optimizer u^* for (B.5) necessarily satisfies

$$\partial_u H(Z, \mathcal{U}^{P,c,u^*}, \Gamma^{u^*}, \gamma, \widetilde{\mathcal{U}}^{u^*}, \widetilde{\Gamma}^{u^*}, u^*) = 0.$$

Using $\gamma_t = e^{-\delta t} Z_t$ and $\widetilde{\Gamma}_t^{u^*} = e^{-\delta t} \overline{\Gamma}_t^{u^*}$, the previous equation is reduced to

$$u_t^* = \frac{\theta^P (\Gamma_t^{u^*} + \overline{\Gamma}_t^{u^*})}{\Psi_t} Z_{t\wedge\tau}^{1-\eta}, \tag{B.7}$$

which is transformed to (B.2) with $E_t = \frac{\overline{\Gamma}_t^{u^*}}{\Gamma_t^{u^*}}$. Plugging (B.2) into (B.4), we obtain (B.3). □

Proof of Propositions 1 and 2

We prove Proposition 1 and the proof of Proposition 2 is similar. The first result is a direct consequence of (14). The second result is a consequence of the form of Z in (9).

Proof of Propositions 3 and 6

We will prove the statement for Proposition 6 and Proposition 3 is then a special case. We will prove the statement for f^P. The statement for f^O can be proven similarly.

When $I_t = P$, recall $\nu = \inf\{s \geq t : I_s \neq I_t\}$, where I is a continuous time Markov process with the intensity $\lambda_t^O = \Lambda^O(Z_t^{1-\eta})$ jumping from P to O. For any $\mathbb{U}$, because $B^{\mathbb{B}}$ is independent of I, it follows from the Girsanov theorem for point processes (see e.g. Jacod (1975)) that

$$\mathbb{P}^{\mathbb{U}}\big[\nu > s \big| \mathcal{F}_s\big] = \beta^P(t,s), \quad \text{where } \beta^P(t,s) = \exp\Big(-\int_t^s \lambda_u^O du\Big).$$

$$\mathbb{E}_t^{\mathbb{U}}\Big[V_\nu^O 1_{\{\nu \leq T\}}\Big] = \mathbb{E}_t^{\mathbb{U}}\Big[\int_t^T \beta^P(t,s) \lambda_s^O V_s^O ds\Big].$$

In the equations above, $\beta^P(t,s)$ is the probability that the state remains at P between t and s, and $\beta^P(t,s)\lambda_s^O ds$ can be thought of as the probability that the state transitions to O between

s and $s+ds$. Using the previous two equations, we transform (A.2) to

$$V_t^P = \sup_{\pi,c}\inf_u \mathbb{E}_t^{\mathbb{U}}\Big[\int_t^T \beta^P(t,s)\Big[F(c_s,V_s^P)+\frac{1-\gamma}{2\theta^P}V_s^P Z_{s\wedge\tau}^{\eta-1}|u_s|^2\Big]ds \\ + \beta^P(t,T)\epsilon U(c_T)+\int_t^T \beta^P(t,s)\lambda_s^O V_s^O ds\Big]. \tag{B.8}$$

Recall $x_t=\log Z_t$. Because of (9) the dynamics of x follow

$$dx_t = -\tfrac{1}{2}|u_t|^2dt - u_t' dB_t^{\mathbb{B}} = \tfrac{1}{2}|u_t|^2dt - u_t' dB_t^{\mathbb{U}}. \tag{B.9}$$

The stopping time τ is reformulated as $\tau=\inf\{t\geq 0: x_t\leq \underline{x} \text{ or } x_t\geq \overline{x}\}$, where $\underline{x}=\log\underline{z}$ and $\overline{x}=\log\overline{z}$. Dynamic programming implies that

$$\tilde{V}_t = \beta^P(0,t)V_t^P+\int_0^t \beta^P(0,s)\Big[F(c_s,V_s^P)+\frac{1-\gamma}{2\theta^P}V_s^P e^{(\eta-1)x_s}|u_s|^2+\lambda_s^O V_s^O\Big]\,ds,$$

is a martingale under $\mathbb{U}$ when u,π,c are agent's optimal strategy and $t<\tau$. To calculate the drift of $\tilde{V}$, we use equation (B.9) and apply Itô's formula to derive

$$de^{f(t,x_t)} = e^{f(t,x_t)}\Big[\partial_t f+\tfrac{1}{2}|u_t|^2\Big(\partial_{xx}^2 f+\partial_x f+(\partial_x f)^2\Big)\Big]dt - e^{f(t,x_t)}\partial_x f u_t' dB_t^{\mathbb{U}}.$$

Moreover define $\tilde{c}=\frac{c}{W}$ as the consumption-wealth ratio. Then

$$d\frac{W_t^{1-\gamma}}{1-\gamma} = W_t^{1-\gamma}\big[r+\pi'(\mu-r-\sigma u)-\tilde{c}-\tfrac{1}{2}\gamma\pi'\Sigma\pi\big]dt + W_t^{1-\gamma}\pi'\sigma dB_t^{\mathbb{U}}, \tag{B.10}$$

Combining the previous two equations, we obtain the drift of $\tilde{V}$ (divided throughout by $\beta^P(0,t)W^{1-\gamma}e^f(t,x_t)$)

$$r+\pi'(\mu-r-\sigma u)-\tilde{c}-\tfrac{1}{2}\gamma\pi'\Sigma\pi+\tfrac{1}{1-\gamma}\big[\partial_t f^P+\tfrac{1}{2}|u|^2(\partial_{xx}^2 f^P+\partial_x f^P+(\partial_x f^P)^2)\big]-\partial_x f^P\pi'\sigma u \\ +\tfrac{\delta}{1-\frac{1}{\psi}}\tilde{c}^{1-\frac{1}{\psi}}e^{-\frac{1}{\nu}f^P}-\delta\tfrac{\nu}{1-\gamma}+\tfrac{1}{2\theta^P}e^{(\eta-1)x}|u|^2+\tfrac{\lambda^O}{1-\gamma}\big(e^{f^O-f^P}-1\big).$$

This drift needs to be nonpositive for arbitrary (u,π,c) and zero for the optimal ones. Therefore, when $\gamma\in(0,1)$, the HJB equation for f^P is

$$0=\sup_{\pi,\tilde{c}}\inf_u\Big\{\partial_t f^P+\tfrac{1}{2}|u|^2\Big(\partial_{xx}^2 f^P+\partial_x f^P+(\partial_x f^P)^2\Big)-(1-\gamma)\partial_x f^P\pi'\sigma u+\delta\nu\tilde{c}^{1-\frac{1}{\psi}}e^{-\frac{1}{\nu}f^P} \\ +(1-\gamma)\big[r+\pi'(\mu-r-\sigma u)-\tilde{c}-\tfrac{1}{2}\gamma\pi'\Sigma\pi\big]-\delta\nu+\tfrac{1-\gamma}{2\theta^P}e^{(\eta-1)x}|u|^2+\lambda^O\big(e^{f^O-f^P}-1\big)\Big\}. \tag{B.11}$$

The supremum and infimum change to $\inf_{\pi,\tilde{c}}\sup_u$ in the previous equation when $\gamma>1$.

The first order condition in u yields

$$u^{P,*} = \frac{(1-\gamma)(1+\partial_x f^P)}{\partial^2_{xx} f^P + \partial_x f^P + (\partial_x f^P)^2 + \frac{1-\gamma}{\theta^P} e^{(\eta-1)x}} \sigma' \pi^{P,*}.$$

This is the agent's optimal belief choice if $\partial^2_{xx} f^P + \partial_x f^P + (\partial_x f^P)^2 + \frac{1-\gamma}{\theta^P} e^{(\eta-1)x} > 0$. Plugging the previous expression of $u^{P,*}$ into (B.11), the first-order condition for π yields

$$\pi^* = \frac{1}{\gamma^{\text{eff},P}} \Sigma^{-1}(\mu - r).$$

This is the agent's optimal strategy when $\gamma^{\text{eff},P}\Sigma$ is positive definite. The agent's optimal choice of consumption wealth is

$$\tilde{c}^{P,*} = \delta^\psi e^{-\frac{\psi}{\nu} f^P}.$$

When the state variable x reaches the boundaries $\underline{x}$ and $\overline{x}$, x is absorbed there, and the problem becomes one where the Cressie-Read penalty in (A.2) is

$$\tfrac{1-\gamma}{2\theta^P} V^P_s e^{(\eta-1)x_\tau} |u_s|^2, \quad \text{for } s \geq \tau.$$

Effectively, this is an entropy penalty

$$\tfrac{1-\gamma}{2\theta^P(x_\tau)} V^P_s |u_s|^2, \quad \text{where } \theta^P(x_\tau) = \theta e^{(1-\eta)x_\tau}.$$

However, the Markov process I can still transition to the state O with the intensity $\Lambda^O(e^{(1-\eta)x_\tau})$. As a result, the boundary conditions of f^P at $\underline{x}$ and $\overline{x}$ are specified by the value $f^{\text{ent},P}_x$ with the robust parameter $\theta^P(\underline{x})$ or $\theta^P(\overline{x})$. Setting the spatial derivatives to be zero in (A.4), $f^{\text{ent},P}_x$, with $x = \underline{x}$ or $\overline{x}$, satisfies the following ODE,

$$\begin{aligned}
0 =& \partial_t f^{\text{ent},P}_x - \delta\nu + (1-\gamma)\big[r + (\pi^{\text{ent},P})'(\mu - r - \sigma u^{\text{ent},P}) - \tfrac{1}{2}\gamma(\pi^{\text{ent},P})'\Sigma\pi^{\text{ent},P}\big] \\
&+ \delta^\psi \tfrac{\nu}{\psi} e^{-\frac{\psi}{\nu} f^{\text{ent},P}_x} + \tfrac{1-\gamma}{2\theta^P} e^{(\eta-1)x} |u^{\text{ent},P}|^2 + \Lambda^O(e^{(1-\eta)x})\big(e^{f^{\text{ent},O}_x - f^{\text{eff},P}_x} - 1\big), \qquad \text{(B.12)} \\
f^{\text{ent},P}_x(T) =& \log \epsilon,
\end{aligned}$$

where

$$\pi^{\text{ent},P} = \frac{1}{\gamma + \theta^P(x)} \Sigma^{-1}(\mu - r), \quad u^{\text{ent},P} = \frac{\theta^P(x)}{\gamma + \theta^P(x)} \sigma' \Sigma^{-1}(\mu - r),$$

and $f^{\text{ent},O}_x$ satisfies a similar ODE coupled with $f^{\text{ent},P}_x$.

Proof of Propositions 4 and 7

When the agent invests in the Lucas tree, her optimal consumption and investment problem can be solved as in Section 2. Instead of constants μ, σ and r, μ^L, σ^L and r depend on the state variable x. However, when the portfolio choice problem is solved in Proposition 6, state variable x is already taken into account. Therefore, even though μ^L, σ^L, and r are now random, no more state variable needs to be introduced and the function f^P in (29) still solves (A.4) with μ and σ therein replaced by μ^L and σ^L. The optimal belief and strategies are given by (A.7), (A.8), and (A.9), when the agent's sentiment state is pessimistic.

We prove the statement when $I = P$. The proof for the $I = O$ case is similar. From consumption market clearing and (A.9),

$$c_t^{P,*} = \delta^\psi e^{-\frac{\psi}{\nu} f^P(t,x_t)} W_t. \tag{B.13}$$

Applying Itô's formula on the right-hand side, yields

$$de^{-\frac{\psi}{\nu} f^P(t,x_t)} = -\frac{\psi}{\nu} e^{-\frac{\psi}{\nu} f^P(t,x_t)} \Big[\partial_t f^P + \tfrac{1}{2}|u|^2\big(\partial^2_{xx} f^P - \partial_x f^P - \tfrac{\psi}{\nu}(\partial_x f^P)^2\big)\Big] dt + \tfrac{\psi}{\nu} e^{-\frac{\psi}{\nu} f^P(t,x_t)} \partial_x f^P u \, dB_t^{\mathbb{B}}.$$

Then using capital market clearing $\pi^* = 1$ and (A.9), we obtain

$$\begin{aligned} de^{-\frac{\psi}{\nu} f^P(t,x_t)} W_t =& W_t de^{-\frac{\psi}{\nu} f^P(t,x_t)} + e^{-\frac{\psi}{\nu} f^P(t,x_t)} dW_t + d\langle e^{-\frac{\psi}{\nu} f^P(t,x_t)}, W_t\rangle_t \\ =& e^{-\frac{\psi}{\nu} f^P} W_t \Big[-\tfrac{\psi}{\nu}\partial_t f^P - \tfrac{\psi}{2\nu}|u|^2\Big(\partial^2_{xx} f^P - \partial_x f^P - \tfrac{\psi}{\nu}(\partial_x f^P)^2\Big)\Big] dt \\ &+ e^{-\frac{\psi}{\nu} f^P} W_t(\mu^L - \delta^\psi e^{-\frac{\psi}{\nu} f^P}) dt + e^{-\frac{\psi}{\nu} f^P} W_t \tfrac{\psi}{\nu} \partial_x f^P u \sigma^L \, dt \\ &+ e^{-\frac{\psi}{\nu} f^P} W_t \big[\tfrac{\psi}{\nu}\partial_x f^P u + \sigma^L\big] dB_t^{\mathbb{B}}. \end{aligned}$$

Using the previous dynamics and matching the drift and volatility on both sides of (B.13), we find

$$\mu^{L,P} = \mu^c + \tfrac{\psi}{\nu}\partial_t f^P + \tfrac{\psi}{2\nu}|u|^2\Big(\partial^2_{xx} f^P - \partial_x f^P - \tfrac{\psi}{\nu}(\partial_x f^P)^2\Big) + \delta^\psi e^{-\frac{\psi}{\nu} f^P} - \tfrac{\psi}{\nu}\partial_x f^P u \sigma^{L,P}, \tag{B.14}$$

$$\sigma^{L,P} = \sigma^c - \tfrac{\psi}{\nu}\partial_x f^P u. \tag{B.15}$$

Plugging (B.15) into the right-hand side of (B.14), we transform $\mu^{L,P}$ into

$$\mu^{L,P} = \mu^c + \tfrac{\psi}{\nu}\partial_t f^P + \tfrac{\psi}{2\nu}|u|^2\Big(\partial^2_{xx} f^P - \partial_x f^P + \tfrac{\psi}{\nu}(\partial_x f^P)^2\Big) + \delta^\psi e^{-\frac{\psi}{\nu} f^P} - \tfrac{\psi}{\nu}\partial_x f^P u \sigma^c \tag{B.16}$$

Combining (A.7) and (B.16), we obtain from capital market clearing that

$$\mu^{L,P} - r^P = \gamma^{\text{eff},P}(\sigma^{L,P})^2. \tag{B.17}$$

Plugging (B.15) and (B.16) back into (A.4) and simplifying, we get (A.10). The expression for u^* in (A.12) is obtained by plugging (B.15) into (A.8) and solving for u^*. Finally, (39) follows from combining (A.10), (B.16) and (B.17).

When x reaches the boundary $\underline{x}$ or $\overline{x}$, f^P is specified by $F_x^{\text{ent},P}$ which is the value function in an equilibrium with an entropy cost. $F_x^{\text{ent},P}$ satisfies (A.13), which is obtained from (A.10) by setting all spatial derivatives to be zero.

Proof of Proposition 5

To find its equilibrium (shadow) price, we first identify the state price density M for the representative agent. Suppose that M follows the dynamics

$$\frac{dM_t}{M_t} = -r_t^I dt - \xi_t^I dB_t^{\mathbb{U}}, \quad M_0 = 1, \tag{B.18}$$

where r^I is the equilibrium risk-free rate in the Lucas tree economy when the representative agent is in the sentiment state $I \in \{O, P\}$. Because the shocks driving the regime switching are independent of fundamental shocks, they are not priced. The market price of risk ξ^I is given by

$$\xi_t^I = \lambda_t^I - u_t^{I,*}. \tag{B.19}$$

where $\lambda_t^I = \frac{\mu_t^{L,I} - r_t^I}{\sigma_t^{L,I}}$ is the equilibrium Sharpe ratio of the Lucas tree. Combining (B.18) and (B.19),

$$M_t = e^{-\int_0^t r_s^I ds} \mathcal{E}\Big(-\int \big(\lambda_s^I - u_s^{I,*}\big) dB_s^{\mathbb{U}} \Big)_t. \tag{B.20}$$

[24]

Define the risk-neutral measure $\mathbb{Q}$ via

$$\frac{d\mathbb{Q}}{d\mathbb{B}}\Big|_{\mathcal{F}_T} = \mathcal{E}\Big(-\int \lambda_s^I dB_s^{\mathbb{B}} \Big)_T.$$

The stock is priced as

$$S_t = \frac{1}{M_t} \mathbb{E}_t^{\mathbb{U}}\Big[\int_t^T M_s D_s ds \Big] = \mathbb{E}_t^{\mathbb{Q}}\Big[\int_t^T e^{-\int_t^s r_v^I dv} D_s ds \Big]. \tag{B.21}$$

It follows from (B.21) that

$$S_t = \mathbb{E}_t^{\mathbb{Q}}\Big[\int_t^{T\wedge\nu} e^{-\int_t^s r_v^I dv} D_s ds + e^{-\int_t^\nu r_\nu^I} S_\nu 1_{\{\nu \le T\}} \Big],$$

where $\nu = \inf\{s \ge t \,|\, I_s \ne I_t\}$ is the next state switching time after t. Using the same argument leading to (B.8), we obtain from the previous equation that

$$S_t = \mathbb{E}_t^{\mathbb{Q}}\Big[\int_t^T \beta^I(t,u) e^{-\int_t^u r_v^I dv} \Big(D_u + \lambda_u^{\overline{I}} S_u \Big) du \Big], \tag{B.22}$$

where $\beta^I(t,u) = \exp\big(-\int_t^u \lambda_v^{\overline{I}} dv \big)$ and $\overline{I} = \{O, P\} \setminus I$. Let $\ell^I = S/D$ be the price-dividend ratio

[24] Here $\mathcal{E}\big(-\int \xi_s dB_s^{\mathbb{U}} \big)_t = \exp\big(-\int_0^t \frac{1}{2} |\xi_s|^2 ds - \int_0^t \xi_s dB_s^{\mathbb{U}} \big)$ is a stochastic exponential.

when the agent's sentiment state is I. It follows from (B.22) that

$$\widetilde{S}_t = e^{-\int_0^t (r_v^I + \lambda_v^{\overline{I}})dv} D_t \ell^I(t, x_t) + \int_0^t e^{-\int_0^s (r_v^I + \lambda_v^{\overline{I}})dv} \Big(D_s + \lambda_s^{\overline{I}} D_s \ell^{\overline{I}}(s, x_s) \Big) ds$$

is a $\mathbb{Q}$-martingale.

Given that the dynamics of x and D under $\mathbb{Q}$ are

$$\begin{aligned} dx_t =& \big(-\tfrac{1}{2}|u_t^{I,*}|^2 + u_t^{I,*}\lambda_t^I \big) dt - u_t^{I,*} dB_t^{\mathbb{Q}}, \\ \frac{dD_t}{D_t} =& \big(\mu^D - \rho\sigma^D \lambda_t^I \big) dt + \sigma^D (\rho dB_t^{\mathbb{Q}} + \sqrt{1-\rho^2} dB_t^{\perp}), \end{aligned}$$

where $B^{\mathbb{Q}}$, defined via

$$B_t^{\mathbb{Q}} = B_t^{\mathbb{U}} + \int_0^t \lambda_s^I - u_s^{I,*} ds = B_t^{\mathbb{B}} + \int_0^t \lambda_s^I ds,$$

is a Brownian motion under $\mathbb{Q}$. Equating the drift of $\widetilde{S}$ to be zero, we get equation (B.23) for ℓ^I:

$$\partial_t \ell^I + \tfrac{1}{2}|u^{I,*}|^2 \partial_{xx}^2 \ell^I + \Big(-\tfrac{1}{2}|u^*|^2 + u^{I,*}\lambda^I - \rho\sigma^D u^{I,*} \Big) \partial_x \ell^I + \big(\mu^D - \rho\sigma^D \lambda^I - r^I \big) \ell^I + 1 + \Lambda^{\overline{I}} \big(\ell^{\overline{I}} - \ell^I \big) = 0, \tag{B.23}$$

with the terminal condition

$$\ell^I(T, \cdot) \equiv 0. \tag{B.24}$$

Note that the previous equation is a pair of coupled equations for ℓ^O and ℓ^P.

To obtain $\mu^{S,I}$ and $\sigma^{S,I}$, we apply Itô's formula to $S_t = D_t \ell^I(t, x_t)$ to obtain

$$\begin{aligned} dS_t = d\big(D_t \ell^I(t, x_t) \big) =& D_t \ell^I(t, x_t) \Big(\frac{\partial_t \ell^I}{\ell^I} - \frac{1}{2}|u^{I,*}|^2 \frac{\partial_x \ell^I}{\ell^I} + \frac{1}{2}|u^{I,*}|^2 \frac{\partial_{xx}^2 \ell^I}{\ell^I} + \mu^D - u^{I,*}\sigma^D \rho \frac{\partial_x \ell^I}{\ell^I} \Big) dt \\ &+ D_t \ell^I(t, x_t) \Big[\Big(-u^{I,*} \frac{\partial_x \ell^I}{\ell^I} + \sigma^D \rho \Big) dB_t^{\mathbb{B}} + \sigma^D \sqrt{1-\rho^2} dB_t^{\perp} \Big]. \end{aligned}$$

Adding $D_t dt$ on both sides and dividing by $S_t = D_t \ell$, we obtain

$$\begin{aligned} \frac{dS_t + D_t dt}{S_t} =& \Big(\frac{\partial_t \ell^I}{\ell^I} - \frac{1}{2}|u^{I,*}|^2 \frac{\partial_x \ell^I}{\ell^I} + \frac{1}{2}|u^{I,*}|^2 \frac{\partial_{xx}^2 \ell^I}{\ell^I} + \mu^D - u^{I,*}\sigma^D \rho \frac{\partial_x \ell^I}{\ell^I} + \frac{1}{\ell^I} \Big) dt \\ &+ \Big(-u^{I,*} \frac{\partial_x \ell^I}{\ell^I} + \sigma^D \rho \Big) dB_t^{\mathbb{B}} + \sigma^D \sqrt{1-\rho^2} dB_t^{\perp}. \end{aligned}$$

Matching the previous equation with equation (42), we obtain $\mu^{S,I}$, $\sigma^{S,I}$, and $\sigma^{S,I,\perp}$. Finally, to obtain the CAPM relation, we note that $S_t M_t + \int_0^t M_s D_s ds$ is a $\mathbb{P}^{\mathbb{U}}$-martingale. Then the CAPM relation follows from combining (B.18), (B.19), and (42).

Numerical procedure

We solve (A.4), (A.5), (B.12) in Proposition 6, (A.10), (A.11), (A.13) in Proposition 7, and (B.23), (B.24) in Proposition 5 using finite difference implicit schemes. For the coupled equations between pessimistic and optimistic utilities, we fix the optimistic utility, and numerically solve the pessimistic utility. Then using the obtained pessimistic utility, we numerically solve the optimistic utility. We iterate until convergence.

Appendix C Log utility

We study the portfolio choice problem for an agent with log utility by taking the scaling limit of Proposition 6 as $\gamma \to 1$. We focus on the pessimistic utility case without regime switching. The optimistic utility case can be analyzed similarly and the regime-switching case can be developed.

Proposition 8. *Consider an agent with the pessimistic utility described in* (15)*, where the preference towards intertemporal consumption is logarithmic. The optimal portfolio choice and the worse-case belief distortion is given by*

$$\pi = \frac{\partial^2_{xx} h + \partial_x h + \frac{1}{\theta^P} e^{(\eta-1)x}}{1 + \partial^2_{xx} h + \partial_x h + \frac{1}{\theta^P} e^{(\eta-1)x}} \Sigma^{-1}(\mu - r), \tag{C.1}$$

$$u = \frac{1}{\partial^2_{xx} h + \partial_x h + \frac{1}{\theta^P} e^{(\eta-1)x}} \sigma' \pi. \tag{C.2}$$

where h satisfies

$$0 = \partial_t h + \tfrac{1}{2}|u|^2 \left(\partial^2_{xx} h + \partial_x h\right) - \delta h - \delta + \delta \log \delta + r + \pi'(\mu - r - \sigma u) - \tfrac{1}{2}\pi' \Sigma \pi + \tfrac{1}{2\theta} e^{(\eta-1)x} |u|^2 = 0, \tag{C.3}$$

with the terminal condition $h(x,T) = 0$.

From (C.1) and (C.2), we observe that the portfolio choice for the log utility agent is still dynamic. The log utility agent still hedges against the future belief variation. This can be also seen from the following identity,

$$\sigma u + \Sigma \pi = \mu - r,$$

which is obtained after combining (C.1) and (C.2). The variation of u drives the variation of π so that the sum of σu and $\Sigma \pi$ is always constant.

Proof. For the function f^P in (29), define h via

$$h(t,x) = \frac{f^P(t,x)}{1-\gamma}.$$

Then the agent's value function (after adding a constant $-\frac{1}{1-\gamma}$) is

$$V = \frac{W^{1-\gamma}e^{(1-\gamma)h} - 1}{1-\gamma} = \frac{e^{(1-\gamma)(\log W + h)} - 1}{1-\gamma}.$$

As $\gamma \to 1$, the right-hand side above converges to $\log W + h$, which is the form of value function in the log utility case with entropy cost in Maenhout (2004), Appendix B.

From (A.4) with $\Lambda^O \equiv 0$ and $\nu = 1$, we derive the equation satisfied by h:

$$\begin{aligned} 0 =&\partial_t h + \tfrac{1}{2}|u|^2\Big(\partial^2_{xx}h + \partial_x h + (1-\gamma)(\partial_x h)^2\Big) - \tfrac{\delta}{1-\gamma} - (1-\gamma)\partial_x h \pi'\sigma u \\ &+ [r + \pi'(\mu - r - \sigma u) - \tfrac{1}{2}\gamma\pi'\Sigma\pi] + \tfrac{\gamma}{1-\gamma}\delta^{\frac{1}{\gamma}} e^{-\frac{1-\gamma}{\gamma}h} + \tfrac{1}{2\theta^P}e^{(\eta-1)x}|u|^2 = 0, \end{aligned} \tag{C.4}$$

where

$$\pi = \Big(\gamma + \frac{(1+(1-\gamma)\partial_x h)^2}{\partial^2_{xx}h + \partial_x h + (1-\gamma)(\partial_x h)^2 + \frac{1}{\theta^P}e^{(\eta-1)x}}\Big)^{-1}\Sigma^{-1}(\mu - r), \tag{C.5}$$

$$u = \frac{1+(1-\gamma)\partial_x h}{\partial^2_{xx}h + \partial_x h + (1-\gamma)(\partial_x h)^2 + \frac{1}{\theta^P}e^{(\eta-1)x}}\sigma'\pi. \tag{C.6}$$

Here we consider $\epsilon = 1$ in the bequest utility. Then the terminal condition for h is $h(T,x) = 0$.

Now send $\gamma \to 1$ in (C.4), (C.5), and (C.6) to identify the limiting equation satisfied by h. We first consider the limit of $-\frac{\delta}{1-\gamma} + \frac{\gamma}{1-\gamma}\delta^{\frac{1}{\gamma}} e^{-\frac{1-\gamma}{\gamma}h}$. When $\gamma \to 1$, up to the first order of $1-\gamma$,

$$-\tfrac{\delta}{1-\gamma} + \tfrac{\gamma}{1-\gamma}\delta^{\frac{1}{\gamma}} e^{-\frac{1-\gamma}{\gamma}h} \approx \frac{-\delta + \gamma\delta^{\frac{1}{\gamma}}(1 - \frac{1-\gamma}{\gamma}h))}{1-\gamma} = \frac{\delta^{\frac{1}{\gamma}} - \delta}{1-\gamma} - \delta^{\frac{1}{\gamma}} - \delta^{\frac{1}{\gamma}}h.$$

By L'Hospital rule, $\lim_{\gamma\to 1}\frac{\delta^{\frac{1}{\gamma}} - \delta}{1-\gamma} = \delta\log\delta$. Then

$$\lim_{\gamma\to 1} -\tfrac{\delta}{1-\gamma} + \tfrac{\gamma}{1-\gamma}\delta^{\frac{1}{\gamma}} e^{-\frac{1-\gamma}{\gamma}h} = -\delta - \delta h + \delta\log\delta.$$

Using the previous identity, we obtain the limit of (C.4), (C.5), and (C.6) as (C.3), (C.1), and (C.2). □